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Restrictions on classical distance-regular graphs

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Abstract Let Γ be a distance-regular graph with diameter $d \geq 2$. It is said to have *classical parameters* (d, b, α, β) when its intersection array $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$ satisfies

$$b_i = ([d]_b - [i]_b)(\beta - \alpha[i]_b) \qquad \text{and} \qquad c_{i+1} = [i+1]_b(1 + \alpha[i]_b) \qquad (0 \leq i \leq d-1),$$

where $[i]_b := 1 + b + \dots + b^{i-1}$. Apart from the well-known families, there are many sets of classical parameters for which the existence of a corresponding graph is still open. It turns out that in most such cases we have either $\alpha = b-1$ or $\alpha = b$. For these two cases we derive bounds on the parameter β, which give us complete classifications when $b = -2$. Distance-regular graphs with classical parameters are antipodal iff $b = 1$ and $\beta = 1 + \alpha[d-1]_b$. If we drop the condition $b = 1$, it turns out that one obtains either bipartite or tight graphs. For the latter graphs, we find closed formulas for the parameters of the CAB partitions and the distance partition corresponding to an edge. Finally, we find a two-parameter family of feasible intersection arrays for tight distance-regular graphs with classical parameters $(d, b, b-1, b^{d-1})$ (primitive iff $b \neq 1$) and apply our results to show that it is realized only by d-cubes ($b = 1$).

Keywords distance-regular graphs · classical parameters · formally self-dual · tight graphs · locally strongly regular

1 Introduction

A general distance-regular graph Γ with diameter d is parameterized with $2d - 1$ parameters, which are usually gathered in the intersection array $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, where $c_1 = 1$ and $k = b_0$ is the valency of the graph. On the other hand, the largest known infinite families of distance-regular graphs (in terms of the number of independent unbounded parameters) are the families of bilinear forms graphs $H_q(d, e)$ ($e \geq d$) and Grassmann graphs $J_q(e, d)$ ($e \geq 2d$). In these two cases, only 3 parameters are needed, so it seems that one could reduce the number of parameters considerably. Indeed, Leonard [17] succeeded to parametrize Q-polynomial distance-regular graphs with only 5 parameters,

$$d, \quad k, \quad c_d, \quad b := b_1/(1+\theta) \quad \text{and} \quad b' := b_2/(\theta - k + b_1 + c_2 - b),$$

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where $\theta = \theta_1$ is the second eigenvalue in the Q-polynomial ordering, cf. [3, Proposition 8.1.5]. Moreover, in 1989, Brouwer, Cohen and Neumaier [3] introduced distance-regular graphs with classical parameters (d, b, α, β), which coincide precisely with Q-polynomial distance-regular graphs having $b = b'$. We give their formal definition and basic properties in Section 3, following the preliminaries. Most known infinite families of distance-regular graphs, including the aforementioned 3-parametric families, can be parameterized in terms of classical parameters. While the special case $b = 1$ has been successfully classified by classical parameters, in particular, the Hamming graphs $H(d, n)$ $(n \neq 4)$, the Johnson graphs $J(n, d)$ $((n, d) \neq (8, 2))$ and the halved n-cubes are characterized by their intersection array [3, Theorem 6.1.1], very much is open in general.

Assuming also $\alpha = b - 1$, we get down to 3 parameters and the formally self-dual case. It is an open problem to find out whether for $b \neq 1$, all graphs with these parameters are already known. They include the bilinear forms graphs $(\beta = b^e - 1)$, which are uniquely determined by their parameters when $b = 2$ and $\beta \geq 2^{d+4} - 1$ or $b \geq 3$ and $\beta \geq b^{d+3} - 1$, see Metsch [19]. Further examples include three 2-parameter families (the alternating, quadratic and Hermitean forms graphs), one 1-parameter family (the affine $E_6(q)$ graphs), and the coset graph of the extended ternary Golay code. For $d \geq 3$, Ivanov and Shpectorov [11,12] and Terwilliger [24] characterized the Hermitean forms graphs by their classical parameters.

Feasibility conditions indicate that for diameter $d \geq 4$ and small number of vertices, the only remaining cases (besides $\alpha = b - 1$) have $\alpha = b$. This case includes the Grassmann graphs $(\beta = b[e - d]_b)$, which are uniquely determined by their parameters when $b = 2$ and $e \geq 2d + 4$, $b = 3$ and $e \geq 2d + 3$, or $b \geq 4$ and $e \geq 2d + 2$, see Metsch [18]. For $e = 2d + 1$, another infinite family is known, namely the twisted Grassmann graphs, see Van Dam and Koolen [6]. The other known examples are the generalized hexagon $\mathrm{GH}(2, 8)$ and the Witt graph associated to M_{23} with classical parameters $(3, -2, -2, 6)$ and $(3, -2, -2, 5)$, respectively.

Among feasible intersection arrays for distance-regular graphs with diameter $d = 3$ on at most 2048 vertices, there are also open cases with classical parameters with α distinct from $b - 1$ and b. There are four such feasible open cases, namely when (b, α, β) take values $(3, 1/2, 5)$, $(3, 5/4, 8)$, $(2, 3, 15)$ or $(2, 5, 21)$, while Coolsaet [4] has already ruled out the case $(2, 1/3, 3)$.

Note that all connected strongly regular graphs have classical parameters. Neumaier [20] has shown that given any integer m, the only strongly regular graphs with smallest eigenvalue $\theta^- = -m$ are complete multipartite graphs $K_{n\times m}$, block graphs of orthogonal arrays $\mathrm{OA}(m, n)$ or Steiner systems $S(2, m, n)$, and a finite number of other graphs. The above three families have classical parameters $(2, m-1, n-2, n-1)$, $(2, m-1, m-2, n-1)$ and $\left(2, m-1, m-1, \frac{n-m}{m-1}\right)$, respectively (with $\alpha = b - 1$ and $\alpha = b$ in the last two cases). As the smallest eigenvalue of a distance-regular graph with classical parameters and $b > 0$ is integral and depends only on d and b, we make the following conjecture that generalizes this result.

Conjecture 1 Let d and b be integers with $d \geq 3$ and $b \neq 0, -1$. Then the distance-regular graphs with classical parameters (d, b, α, β) are the Hamming graphs, the Johnson graphs, the bilinear forms graphs, the Grassmann graphs, and a finite number of other graphs.

The above conjecture is further supported by a result of Metsch [19, Corollary 1.3] stating that for a distance-regular graph not belonging to one of the families mentioned in Conjecture 1, the parameter β is bounded in terms of d, b and α. A direct corollary of his result is that there is a finite number of distance-regular graphs with classical parameters (d, b, α, β) for any fixed $d \geq 3$, $b \neq 0, -1$ and α not belonging to one of these families.

Several intersection arrays with classical parameters have been found for which the existence of a corresponding graph is not known. Due to the above information, we focus on the cases

$$\alpha \in \{b - 1, b\}$$

with diameter $d \geq 3$, and derive some lower and upper bounds on the parameter β in Section 4. For $b = -2$, these bounds imply a complete classification (extending the classifications by Weng et al. [26,10] to $d = 3$). We also find a 1-parameter family of feasible intersection arrays for distance-regular graphs with classical parameters $(3, b, b, (b + 1)^2)$, $b \in \mathbb{N}$. For $b = 1$ we have the Johnson graph $J(7, 3)$, while already the next case with 430 vertices is open.

Distance-regular graphs with classical parameters are bipartite iff $\alpha = 0$ and $\beta = 1$, and they are antipodal iff $b = 1$ and $\beta = 1 + \alpha(1 + b + \ldots + b^{d-2})$, see [3, Proposition 6.3.1]. If we drop the condition $b = 1$ in the second case, it turns out that one obtains either bipartite or tight graphs. The latter were introduced and characterized in many ways by Jurišić, Koolen and Terwilliger [15]. In Sections 5 and 6 we study tight graphs with classical parameters and find closed formulas for all the triple intersection numbers introduced by Jurišić and Koolen [13] (corresponding to the CAB and the 1-homogeneous properties) and integral parameters of the locally strongly regular graph. As a byproduct, we offer an alternative proof of the classification of all antipodal distance-regular graphs with classical parameters, that is, we prove that Γ is antipodal with diameter $d \geq 3$ if and only if Γ is the Gosset graph, the d-cube, the halved $(2d)$-cube, or the Johnson graph $J(2d, d)$. Next, we find in Section 7 a two-parameter feasible family of intersection arrays for tight distance-regular graphs with classical parameters $(d, b, b-1, b^{d-1})$, primitive iff $b \neq 1$, and use the above mentioned integrality conditions for the local graph to show that it is realized only by d-cubes $(b = 1)$.

2 Preliminaries

In this section we review some basic definitions and concepts. See Brouwer, Cohen and Neumaier [3] and Godsil [9] for further details.

Let Γ be a finite, undirected, connected graph, without loops or multiple edges, with vertex set $V\Gamma$, edge set $E\Gamma$, shortest path-length distance function ∂, and diameter $d := \max\{\partial(x, y) \mid x, y \in V\Gamma\}$. Given a vertex $u \in V\Gamma$, we define the *local graph* $\Gamma(u)$ to be the graph induced by the neighbours of the vertex u in the graph Γ.

A graph Γ is said to be *regular with valency* k if each vertex of Γ has precisely k neighbours. A regular graph Γ of valency k is said to be *strongly regular* with parameters (v, k, λ, μ) if $v = |V\Gamma|$ and for each pair of distinct vertices $x, y \in V\Gamma$, the number of their common neighbours is λ when they are adjacent and μ when they are not. A graph Γ with diameter d is said to be *antipodal* if the relation of being at distance 0 or d is an equivalence relation on the vertices of Γ. A graph Γ is called *bipartite* if there is a partition of $V\Gamma$ into two nonempty independent sets.

A graph Γ is said to be *distance-regular* when for every pair of vertices $u, v \in V\Gamma$ with $\partial(u, v) = h$, the number p^h_{ij} of vertices w with $\partial(u, w) = i$ and $\partial(v, w) = j$ only depends on h, i, j. We call p^h_{ij} $(0 \leq h, i, j \leq d)$ *intersection numbers*. We define $c_i := p^i_{1,i-1}$, $a_i := p^i_{1i}$, $b_i := p^i_{1,i+1}$, $k_i := p^0_{ii}$ $(0 \leq i \leq d)$, where we have assumed $p^0_{1,-1} = p^d_{1,d+1} = 0$. Moreover, $a_0 = c_0 = b_d = 0$, $c_1 = 1$ and $a_i + b_i + c_i = k$ $(0 \leq i \leq d)$, where $k := k_1$ is the valency of Γ. We call the sequence $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$ the *intersection array* of a distance-regular graph. The intersection array uniquely determines all other intersection numbers. We call an intersection array *feasible* if it has not been proven that a corresponding distance-regular graph cannot exist – in particular, all intersection numbers must be nonnegative integers. Many other feasibility conditions can be found in Brouwer et al. [3]. Non-complete connected strongly regular graphs are precisely the distance-regular graphs with diameter $d = 2$.

Let A_i denote the distance matrix of Γ, i.e., the binary matrix indexed by the vertices of the graph Γ, where $A_i(u, v) = 1$ iff $\partial(u, v) = i$ for any $u, v \in V\Gamma$. The matrix $A := A_1$ is known as the *adjacency matrix*. The *eigenvalues* of Γ are defined as the eigenvalues of the matrix A. Since A is nonnegative and symmetric, the eigenvalues are real numbers. A regular graph with valency k has k as its largest eigenvalue, with multiplicity equal to the number of connected components. A connected regular graph with diameter at least two is strongly regular if and only if it has precisely three distinct eigenvalues. A distance-regular graph with diameter d has precisely $d+1$ distinct eigenvalues [3, p. 128], which are usually denoted in decreasing order as $k = \theta_0 > \theta_1 > \cdots > \theta_d$, called the *natural ordering*. We denote the multiplicity of the eigenvalue θ_i by m_i $(0 \leq i \leq d)$. The integrality of multiplicities turns out to be a very strong feasibility condition.

Let $\mathcal{M}$ be the Bose-Mesner algebra of a distance-regular graph Γ, that is, the subalgebra of $\mathbb{R}^{n \times n}$ generated by A. Then the matrices $\{A_i\}_{i=0}^d$ form a basis of $\mathcal{M}$, which also has a basis $\{E_i\}_{i=0}^d$, where E_i is the projector onto the eigenspace of A associated to the eigenvalue θ_i [3, p. 45]. We define matrices P and Q such that $A_j = \sum_{i=0}^d P_{ij} E_i$ and $E_j = |V\Gamma|^{-1} \sum_{i=0}^d Q_{ij} A_i$ $(0 \leq j \leq d)$, and call them the *eigenmatrix* and *dual eigenmatrix*, respectively. A graph is *formally self-dual* if $P = Q$ holds for some ordering of eigenvalues [3, p. 49]. Furthermore, we can define *Krein parameters* q^h_{ij} [3, p. 48] as such numbers that $E_i \circ E_j = |V\Gamma|^{-1} \sum_{h=0}^d q^h_{ij} E_h$ $(0 \leq i, j \leq d)$, where $\circ$ represents entrywise multiplication of matrices. If Γ is formally self-dual, then $p^h_{ij} = q^h_{ij}$.

Let u, v and w be vertices of a graph Γ such that $\partial(u, v) = W$, $\partial(u, w) = V$ and $\partial(v, w) = U$. Define

$$\begin{bmatrix} u & v & w \\ h & i & j \end{bmatrix} := |\{x \in V\Gamma \mid \partial(u, x) = h, \partial(v, x) = i, \partial(w, x) = j\}|,$$

i.e., it is the number of vertices at distances h, i, j from u, v, w, respectively. These numbers are called *triple intersection numbers*. Unlike the intersection numbers, the triple intersection numbers of a distance-regular graph may depend on the choice of the three vertices and not only on their mutual distances. However, we can derive the following equations:

$$p^W_{hi} = \sum_{s=0}^{d} \begin{bmatrix} u & v & w \\ h & i & s \end{bmatrix}, \quad p^V_{hj} = \sum_{s=0}^{d} \begin{bmatrix} u & v & w \\ h & s & j \end{bmatrix}, \quad p^U_{ij} = \sum_{s=0}^{d} \begin{bmatrix} u & v & w \\ s & i & j \end{bmatrix} \quad (0 \le h,i,j \le d). \tag{1}$$

Additionally, we have

$$\begin{bmatrix} u & v & w \\ h & i & 0 \end{bmatrix} = \delta_{hV}\delta_{iU}, \quad \begin{bmatrix} u & v & w \\ h & 0 & j \end{bmatrix} = \delta_{hW}\delta_{jU}, \quad \begin{bmatrix} u & v & w \\ 0 & i & j \end{bmatrix} = \delta_{iW}\delta_{jV}, \quad (0 \le h,i,j \le d), \tag{2}$$

by which we can interpret (1) as a system of $3d^2$ equations with d^3 nonnegative integral variables. As some of these variables must be zero due to the triangle inequality, it is sometimes possible to express the general solution of the system with a small number of parameters, see for example [5,25,16]. If a Krein parameter q^h_{ij} is zero (we say that it *vanishes*), we can obtain another equation for triple intersection numbers, see [5, Theorem 3], cf. Brouwer et al. [3, Theorem 2.3.2].

Theorem 1 *Let Γ be a distance-regular graph with diameter d, dual eigenmatrix Q, and Krein parameters q^h_{ij} $(0 \le i,j,h \le d)$. For vertices $u,v,w \in V\Gamma$, define a triple sum*

$$S_{ijh}(u,v,w) := \sum_{r,s,t=0}^{d} Q_{ri}Q_{sj}Q_{th} \begin{bmatrix} u & v & w \\ r & s & t \end{bmatrix}.$$

Then $S_{ijh}(u,v,w) = 0$ whenever $q^h_{ij} = 0$. □

Although the Krein parameter q^h_{ij} vanishing is equivalent to the vanishing of any Krein parameter with the indices h,i,j permuted, this may give multiple linearly independent Diophantine equations by the above theorem, see [16]

3 Distance-regular graphs with classical parameters

Let b be a real number and i an integer. Since we will only use the Gaussian binomial coefficient

$$\begin{bmatrix} i \\ 1 \end{bmatrix}_b = \begin{cases} i & \text{if } b = 1, \\ (b^i - 1)/(b-1) & \text{otherwise,} \end{cases}$$

we will denote it simply by $[i]_b$. Note that $[0]_b = 0$, $[1]_b = 1$, $[2]_b = b+1$, and in general, $[i]_b = \sum_{j=0}^{i-1} b^j$ for $i > 0$ and $[i]_b = -\sum_{j=i}^{-1} b^j$ for $i < 0$.

We start with a lemma that will be useful for calculating with the Gaussian binomial coefficients.

Lemma 1 *Let s and t be integers. Then*

$$\text{(i) } [s]_b - [t]_b = b^t[s-t]_b \qquad \textit{and} \qquad \text{(ii) } [t+2]_b + b[t]_b = (b+1)[t+1]_b.$$ □

A distance-regular graph Γ with diameter d is said to have *classical parameters* (d,b,α,β) when $d \ge 2$ and its intersection array satisfies

$$b_i = ([d]_b - [i]_b)(\beta - \alpha[i]_b) \quad \text{and} \quad c_i = [i]_b(1 + \alpha[i-1]_b) \quad (0 \le i \le d). \tag{3}$$

By [3, Proposition 6.2.1], the parameter b is an integer distinct from 0 and -1 when $d \ge 3$.

The following result of Brouwer et al. [3, Corollary 8.4.2, Theorem 8.4.3] will be used to compute the eigenvalues of classical distance-regular graphs and their multiplicities.

Lemma 2 *The eigenvalues* $\{\theta_i\}_{i=0}^d$ *of a distance-regular graph with classical parameters* (d, b, α, β) *are determined by the relations*

$$\theta_i = [d-i]_b\,(\beta - \alpha[i]_b) - [i]_b.$$

If b *is positive, then the eigenvalues are given in the natural ordering.*
The multiplicities $\{m_i\}_{i=0}^d$ *are determined by the relations*

$$m_i = \frac{\left(1+\alpha[d-2i]_b + b^{d-2i}\beta\right)\prod_{j=0}^{i-1}\alpha_j}{\left(1+\alpha[d]_b + b^d\beta\right)\prod_{j=1}^{i}\beta_j},$$

where

$$\alpha_j = b[d-j]_b\,(\beta - \alpha[j]_b)\left(1+\alpha[d-j]_b + b^{d-j}\beta\right) \qquad (0 \le j \le d-1),$$
$$\beta_j = [j]_b\left(\beta - \alpha[j]_b + b^j\right)\left(1+\alpha[d-j]_b\right) \qquad (1 \le j \le d). \qquad \square$$

We have already mentioned that the parameter b is integral for $d \ge 3$. We can say even more.

Lemma 3 *Let* Γ *be a distance-regular graph with classical parameters and diameter* $d \ge 3$.

(i) If d *is odd, then* β *is integral.*
(ii) If d *is even, then* $\beta - \alpha$ *is integral.*

Proof Since Γ has diameter at least 3, the parameter b is integral. By (3) and Lemma 1(i), we have $k = [d]_b\beta$, $b_1 = ([d]_b - 1)(\beta - \alpha)$ and $c_2 = (b+1)(\alpha+1)$. Therefore, we also have $(b+1)\alpha = c_2 - b - 1$, which implies that $(b+1)\alpha$ is integral, and $\beta = k - b_1 - b\alpha[d-1]_b$. By multiplying the last relation by $(b+1)$, we conclude that $(b+1)\beta$ is integral as well. If d is odd, then $k = [d]_b\beta$ implies $k \equiv \beta \pmod{(b+1)\beta}$ and (i) follows. On the other hand, if d is even, then $b_1 = ([d]_b - 1)(\beta - \alpha)$ implies $b_1 \equiv \alpha - \beta \pmod{(b+1)(\beta-\alpha)}$ and (ii) follows. $\square$

We also review Weng's result [26, Theorem 10.3].

Theorem 2 *Let* Γ *be a distance-regular graph with classical parameters* (d, b, α, β), *where* $d \ge 4$ *and* $b < 0$, *and intersection numbers* $a_1 \ne 0$ *and* $c_2 > 1$. *Then one of the following holds:*

(a) Γ *is the dual polar graph* ${}^2A_{2d-1}(-b)$ $\quad(\alpha = b(b-1)/(b+1), \beta = -b(b^d+1)/(b+1))$,
(b) Γ *is the Hermitean forms graph* $\mathrm{Her}(d, -b)$ $\quad(\alpha = b-1, \beta = -b^d - 1)$,
(c) $\alpha = (b-1)/2$, $\beta = -(1+b^d)/2$, *and* $-b$ *is a power of an odd prime.* $\square$

4 Bounds on the parameter β

Let us first give some general upper and lower bounds on β.

Proposition 1 *Let* Γ *be a distance-regular graph with classical parameters and diameter* $d \ge 3$. *Then the following hold.*

(i) $\beta \ge 1 - \alpha b[d-1]_b$, *with equality if and only if* $a_1 = 0$.
(ii) If $b < 0$, *then* $\beta \le (-[d]_b + b + 2)\alpha$, *with equality if and only if* $b_1 = b_2$.
(iii) If $b > 0$, *then* $\beta \ge 1 + \alpha[d-1]_b$, *with equality if and only if* $a_d = 0$.

Proof (i) By $a_1 \ge 0$, we have $k \ge b_1 + 1$, which is equivalent to the stated bound by (3).

(ii) Using $b_1 \ge b_2$ and (3), we derive $b\beta \ge (-[d]_b + b + 2)b\alpha$, which is equivalent to the stated bound as $b < 0$.

(iii) By $a_d \ge 0$, we have $k \ge c_d$, which is equivalent to $[d]_b\beta \ge [d]_b(1 + \alpha[d-1]_b)$ by (3). As b is positive, $[d]_b$ is also positive and the stated bound then follows. $\square$

There are many open cases for both conditions $\alpha = b - 1$ and $\alpha = b$. For either condition, we derive some bounds for the parameter β. We start with the case when the parameter b is negative. The next two results improve the bounds given by Metsch [19, Corollary 1.3(a)].

Theorem 3 *Let Γ be a distance-regular graph with classical parameters $(d, b, b-1, \beta)$, $d \geq 3$ and $b < 0$. Then $-b^d + b + 1 \leq \beta$, with equality if and only if $a_1 = 0$, and $\beta \leq -b(b+1)(b-2)$ if $d = 3$. If $d \geq 4$, then Γ is a Hermitean forms graph, or $b \neq -2$ and the lower bound is attained, i.e., $a_1 = 0$.*

Proof By Proposition 1(i) we derive $\beta \geq (1-b)b[d-1]_b + 1$, which is equivalent to the stated lower bound. To establish the upper bound, we use Proposition 1(ii) to obtain

$$\beta \leq -b^d + b^2 + b - 1. \tag{4}$$

Let us assume $d = 3$. Since the classical parameter b is an integer distinct from 0 and -1, we have $b \leq -2$. As β is integral by Lemma 3(i), it suffices to show that $t := \beta + b(b+1)(b-2) - 1 < 0$ holds. We will assume the contrary and obtain a contradiction. By Brouwer et al. [3, p. 169(1a)], the parameters of a distance-regular graph with diameter at least 2 satisfy

$$\left\lfloor \frac{a_1(a_1-1)}{b_1} \right\rfloor \geq \min(c_2 - 1, b_2 - b_1 + a_1 + 1). \tag{5}$$

Using (3) and $a_1 = k - b_1 - 1 = b^2 + b + t$, we compute $c_2 = b^2 + b \geq 2$ and $b_2 - b_1 + a_1 + 1 = (1-b)(t+1) > 1$, so the right hand side of (5) is at least 1. By (4), we have $t \leq -b-2$. Assuming $t \geq 0$, this implies, together with $b \leq -2$ and $b^2(1-b) \geq -b-1$ (i.e., $b^2 \geq 1 - 2/(1-b)$), the inequality

$$b_1 = -(b+1)b^2(b^2-b)\left(1 - \frac{-b-2-t}{b^2(1-b)}\right) \geq -(b+1)b^2(b^2-b)\left(1 - \frac{-b-2}{-b-1}\right) = b^2(b^2-b) > a_1(a_1-1). \tag{6}$$

Hence $\lfloor a_1(a_1-1)/b_1 \rfloor = 0$, a contradiction.

For $d \geq 4$, the graph Γ has $c_2 = b(b+1) > 1$. Suppose that Γ is not isomorphic to a Hermitean forms graph. Then we have $a_1 = 0$ by Theorem 2, and β attains the lower bound by Proposition 1(i). As this bound is also attained by Hermitean forms graphs $\mathrm{Her}(d, 2)$, which are determined by their parameters, cf. Ivanov and Shpectorov [11], it follows that we then have $b \neq -2$. □

Theorem 4 *Let Γ be a distance-regular graph with classical parameters (d, b, b, β), $d \geq 3$ and $b < 0$. Then $d = 3$, and β is an integer with*

$$-b^3 - b^2 + 1 \quad \leq \quad \beta \quad \leq \quad -b^3 + b. \tag{7}$$

If the lower bound is attained, then we have $b = -2$.

Proof By Proposition 1(i, ii) and Lemma 1(i) we derive $-b^2[d-1]_b + 1 \leq \beta \leq -b^3[d-2]_b + b$, which is equivalent to the stated bounds in the case $d = 3$.

The graph Γ has $c_2 = (b+1)^2$. If the lower bound for β is attained, then we have $a_1 = 0$ by Proposition 1(i), and also $a_2 = -b(b+1)^2 \neq 0$. By Pan and Weng [22, Theorem 2.1], we then have $c_2 \leq 2$. Since c_2 is a nonzero square, we thus have $c_2 = 1$ and $b = -2$, and then also $\beta = ((-2)^{d+1} - 1)/3$. By Huang, Pan and Weng [10, Theorem 7], such a graph does not exist when $d \geq 4$.

Assume now that $d \geq 4$ holds and the lower bound for β is not attained, so we have $a_1 \neq 0$. If $b = -2$, we have

$$\beta \geq ((-2)^{d+1} - 1)/3 + 1 = (-2)^3((-2)^{d-2} - 1)/3 - 2.$$

The obtained expression is exactly the upper bound and so equality must hold. However, this family has been ruled out by De Bruyn and Vanhove [7, Corollary 1]. On the other hand, $b \neq -2$ implies $c_2 > 1$. We can thus apply Theorem 2, however, none of the possibilities satisfies $\alpha = b$ (note that $b \neq -1$). Therefore, no distance-regular graph with classical parameters with $\alpha = b$ and $d \geq 4$ exists. □

Note that the bounds for $d = 3$ in Theorems 3 and 4 imply that there are at most $b(b+1)$ possibilities for β. In the case $\alpha = b - 1$, all such choices of β give a feasible intersection array. However, this is not true in the case $\alpha = b$, where nonintegral intersection numbers or eigenvalue multiplicities may occur for certain choices of β.

The product $b(b+1)$ is the smallest for $b = -2$, hence we can extend Weng's classification to the case $d = 3$. In this case the following results show that our lower and upper bounds are the best possible.

Corollary 1 *Let Γ be a distance-regular graph with classical parameters $(d, -2, -3, \beta)$ and $d \geq 3$. Then Γ is one of the following:*

(*a*) *the coset graph of the extended ternary Golay code* $(d = 3, \beta = 8)$,
(*b*) *the Hermitean forms graph* $\mathrm{Her}(d, 2)$ $(\beta = -(-2)^d - 1)$.

Proof First, recall that β is integral by Lemma 3. For $d = 3$, Theorem 3 gives $\beta \in \{7, 8\}$, and the corresponding graphs have intersection arrays $\{21, 20, 16; 1, 2, 12\}$ and $\{24, 22, 20; 1, 2, 12\}$. These two cases uniquely determine the Hermitean forms graph $\mathrm{Her}(3, 2)$ and the coset graph of the extended ternary Golay code, respectively, cf. Brouwer et al. [3, p. 427]. For $d \geq 4$, the graph Γ must be isomorphic to the Hermitean forms graph $\mathrm{Her}(d, 2)$ by Theorem 3. □

Corollary 2 *Let Γ be a distance-regular graph with classical parameters $(d, -2, -2, \beta)$ and $d \geq 3$. Then Γ is one of the following:*

(*a*) *the Witt graph associated to M_{23}* $(d = 3, \beta = 5)$,
(*b*) *the generalized hexagon* $\mathrm{GH}(2, 8)$ $(d = 3, \beta = 6)$.

Proof By Theorem 4, we have $d = 3$ and $\beta \in \{5, 6\}$, and the corresponding graphs have intersection arrays $\{15, 14, 12; 1, 1, 9\}$ and $\{18, 16, 16; 1, 1, 9\}$. These two intersection arrays uniquely determine the Witt graph associated to M_{23} and the generalized hexagon $\mathrm{GH}(2, 8)$, respectively, cf. Brouwer et al. [3, pp. 426–427]. □

We improve the bound from Proposition 1(iii) for graphs with diameter three and $\alpha = b - 1 \geq 1$.

Theorem 5 *Let Γ be a distance-regular graph with classical parameters, diameter $d = 3$ and $\alpha = b - 1 \geq 1$. Then*

$$\beta \quad \geq \quad \begin{cases} 6 & \text{if } b = 2, \\ b^2 + \lfloor \sqrt{b + 1} \rfloor & \text{otherwise.} \end{cases}$$

In particular, a distance-regular graph with classical parameters $(b, \beta) = (3, 10)$ or $(8, 66)$, i.e., with intersection array $\{130, 96, 18; 1, 12, 117\}$ or $\{4818, 4248, 192; 1, 72, 4672\}$, does not exist.

Proof First, we derive $b_2 = b^2(\beta - b^2 + 1)$ from (3), from which $\beta \geq b^2$ follows. We will obtain a better bound using $p^3_{33} \geq 0$. We express the intersection number p^3_{33} of Γ in terms of b and β using [3, Lemma 4.1.7]. It turns out that for the cubic polynomial

$$p(x) = x^3 - x^2(2b^2 + 2b - 1) + x(b^4 + 4b^3 - b - 1) - 2b^5 - b^4 + 2b^2 + b - 1,$$

we have $p^3_{33} = p(\beta)$. By solving $p''(x) = 0$ for x, we obtain the inflection point of $p(x)$ at $x = b^2 - (b-1)^2/3 < b^2$. As the leading coefficient of $p(x)$ is positive and $p'(b^2) = (b - 1)(2b + 1) > 0$, it follows that $p(x)$ is increasing on the interval $[b^2, \infty)$. Since $p(b^2) = -(b - 1)(b^2 - 1) < 0$, there is a unique root of $p(x)$ on this interval. Finally,

$$p(b^2 + \sqrt{b} - 1) = -2b(\sqrt{b} - 1)^2 < 0 \quad \text{and} \quad p(b^2 + \sqrt{b}) = (2b^2 - 1)\sqrt{b} - (b - 1)^2 > 0$$

imply that this root must lie on the interval $(b^2 + \sqrt{b} - 1, b^2 + \sqrt{b})$. As $p^3_{33} \geq 0$ and $\beta \in \mathbb{Z}$ by Lemma 3(i), this gives us $\beta \geq b^2 + \lfloor \sqrt{b} \rfloor$. For $b = 2$, the equality case amounts to $\beta = 5$, i.e., intersection array $\{35, 24, 8; 1, 6, 28\}$, which was ruled out by the authors [16]. Hence, $\beta \geq 6$. Now we assume $b \geq 3$. If $b + 1$ is not a perfect square, then we have $\lfloor \sqrt{b} \rfloor = \lfloor \sqrt{b + 1} \rfloor$, which gives the stated lower bound. Otherwise, we have $b = r^2 - 1$ for some integer $r > 1$, and the equality case translates to

$$\beta = (r^2 - 1)^2 + r - 1 = r(r - 1)(r^2 + r - 1) \quad \text{and} \quad p^3_{33} = -r^4 + 4r^3 - 2r^2 - 3r + 1,$$

which is nonnegative only for $r \in \{2, 3\}$. To complete the proof, we thus have to rule out these two cases.

Let Γ be a distance-regular graph with classical parameters $\big(3, r^2 - 1, r^2 - 2, r(r - 1)(r^2 + r - 1)\big)$, where $r \in \{2, 3\}$. As the intersection number p^3_{33} of Γ is 3 or 1 in the cases $r = 2$ and $r = 3$, respectively, we can pick vertices $u, v, w \in V\Gamma$ pairwise at distance 3. Let $[h\ i\ j] := \begin{bmatrix} u & v & w \\ h & i & j \end{bmatrix}$ $(0 \leq h, i, j \leq 3)$. Since the graph Γ is formally self-dual with the natural ordering of the eigenvalues, its eigenmatrices satisfy $P = Q$, and the triangle inequality additionally gives $q^3_{11} = q^1_{13} = q^1_{31} = 0$.

Case $r = 2$. The intersection numbers p^3_{ij} $(0 \le i, j \le 3)$ of Γ and its eigenmatrices are

$$\left(p^3_{ij}\right)_{i,j=0}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 117 & 13 \\ 0 & 117 & 780 & 143 \\ 1 & 13 & 143 & 3 \end{pmatrix}, \quad P = Q = \begin{pmatrix} 1 & 130 & 2040 & 160 \\ 1 & 31 & -16 & -16 \\ 1 & -2 & -5 & 6 \\ 1 & -13 & 39 & -27 \end{pmatrix}.$$

Let $\phi := [1\ 2\ 2]$, $\chi := [2\ 1\ 2]$, $\psi := [2\ 2\ 1]$ and $\xi := [3\ 3\ 3]$. Using (1) and (2), we can express the triple intersection numbers $[h\ i\ j]$ $(0 \le h, i, j \le 3)$ in terms of the these parameters, cf. [16]. As the Krein parameters q^3_{11}, q^1_{13} in q^1_{31} vanish, Theorem 1 adds three new linearly independent equations to the system, which now has general solution $\phi = \chi = \psi = 102 + 3\xi/16$. Finally,

$$0 \le [1\ 3\ 3] = -2 + \frac{3\xi}{16} \quad \text{and} \quad 0 \le [2\ 3\ 3] = 4 - \frac{19\xi}{16},$$

which is equivalent to $64/6 \le \xi \le 64/19$, contradiction. Hence, $r \ne 2$.

Case $r = 3$. The intersection numbers p^3_{ij} $(0 \le i, j \le 3)$ of Γ and its eigenmatrices are

$$\left(p^3_{ij}\right)_{i,j=0}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4672 & 146 \\ 0 & 4672 & 268056 & 11534 \\ 1 & 146 & 11534 & 1 \end{pmatrix}, \quad P = Q = \begin{pmatrix} 1 & 4818 & 284262 & 11682 \\ 1 & 530 & -354 & -177 \\ 1 & -6 & -19 & 24 \\ 1 & -73 & 584 & -512 \end{pmatrix}.$$

As $p^3_{33} = 1$, we also have $[1\ 3\ 3] = [3\ 1\ 3] = [3\ 3\ 1] = [2\ 3\ 3] = [3\ 2\ 3] = [3\ 3\ 2] = [3\ 3\ 3] = 0$. The equations (1) and (2) then give $[1\ 2\ 2] = [2\ 1\ 2] = [2\ 2\ 1] = 4526$, $[1\ 2\ 3] = [2\ 3\ 1] = [3\ 1\ 2] = [1\ 3\ 2] = [2\ 1\ 3] = [3\ 2\ 1] = 146$, $[2\ 2\ 2] = 252142$, and $[2\ 2\ 3] = [2\ 3\ 2] = [3\ 2\ 2] = 11388$. Since the Krein parameter q^3_{11} vanishes, we can derive a new equation by Theorem 1, which cannot be satisfied by the obtained values. Thus, $r \ne 3$. □

Remark. For a distance-regular graph with classical parameters $(3, b, b, \beta)$ and $b \ge 2$ we used similar techniques $(b_2 > 0,\ k_3 \in \mathbb{N})$ to prove $\beta \ge \min\{(b+1)^2, (b+1)^2 - b(1 - \sqrt{2/b}) + 1/2 + 25/(8\sqrt{2b})\}$. We do not know of any examples of feasible intersection arrays with $\beta < (b+1)^2$. However, a family of feasible intersection arrays for distance-regular graphs does arise from the classical parameters

$$\left(3, b, b, (b+1)^2\right).$$

By Theorem 4, we have $b \ge 1$, as β is below the lower bound in (7) when b is negative. For $b = 1$, we have the Johnson graph $J(7, 3)$. No example is known for $b \ge 2$. The smallest open case, i.e., $b = 2$, which has intersection array $\{63, 42, 12; 1, 9, 49\}$ and 430 vertices, is listed by Brouwer et al. [3, p. 430] in the tables of feasible intersection arrays.

5 Tight graphs with classical parameters

Jurišić, Koolen and Terwilliger established the following bound for distance-regular graphs with diameter $d \ge 2$, and characterize the case of equality, see [15] and [3, Theorem 1.3.1].

Theorem 6 *Let Γ be a distance-regular graph of valency k with diameter $d \ge 2$, intersection numbers a_1, b_1 and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then, the following inequality holds:*

$$\left(\theta_1 + \frac{k}{a_1 + 1}\right)\left(\theta_d + \frac{k}{a_1 + 1}\right) \quad \ge \quad -\frac{k a_1 b_1}{(a_1 + 1)^2}. \tag{8}$$

If Γ is non-bipartite, then equality implies that $a_i > 0$ iff $1 \le i \le d - 1$, and the local graph $\Gamma(u)$ $(u \in V\Gamma)$ is connected strongly regular on k vertices with eigenvalues a_1, $\theta^+ = -1 - b_1/(1 + \theta_d)$, $\theta^- = -1 - b_1/(1 + \theta_1)$, parameters $\mu = a_1 + \theta^+\theta^-$, $\lambda = \mu + \theta^+ + \theta^-$, and eigenvalue multiplicities 1, $m^+ = a_1(a_1 - \theta^-)(\theta^- + 1)/(\mu(\theta^- - \theta^+))$, $m^- = a_1(a_1 - \theta^+)(\theta^+ + 1)/(\mu(\theta^+ - \theta^-))$. □

A non-bipartite distance regular graph with diameter at least 2 is called *tight* if equality holds in (8). The following proposition characterizes tight distance-regular graphs with classical parameters.

Proposition 2 *Let Γ be a distance-regular graph with classical parameters. Then*

(i) Γ is tight if and only if $\beta = 1 + \alpha[d-1]_b$ and $b, \alpha > 0$, and
(ii) Γ is antipodal if and only if $b = 1$ and Γ is tight or bipartite.

Proof (i) ($\Longrightarrow$) By Theorem 6, we have $a_i > 0$ iff $1 \le i \le d-1$. By (3), we then have $[d]_b\beta = k = c_d = [d]_b\,(1+\alpha[d-1]_b)$, from which the expression for β follows. Moreover, by Lemma 1(i) we compute

$$a_i = k - b_i - c_i = \alpha[i]_b\,([d]_b + [d-1]_b - [i]_b - [i-1]_b) = \alpha b^{i-1}(b+1)[i]_b[d-i]_b\,. \tag{9}$$

If $b < 0$, then $\operatorname{sign}([i]_b[d-i]_b) = \operatorname{sign}((-1)^d)$ does not depend on i. In the case $d = 2$, $a_1 > 0$ implies $\alpha < 0$, and (3) then gives $b_1 = b < 0$, contradiction. When $d \ge 3$, a_1 and a_2 are not both positive, which again gives a contradiction. Therefore, $b > 0$ holds in either case, and then also $\alpha > 0$ follows from (9).

($\Longleftarrow$) After computing the values k, b_1, a_1, θ_1 and θ_d by (3), (9) and Lemma 2, a straightforward calculation shows that equality holds in (8). As $b, \alpha > 0$, we have $a_1 > 0$, so the graph is not bipartite and is therefore tight.

(ii) Straightforward from (i) and [3, Proposition 6.3.1]. □

Remarks. (i) As a graph Γ with classical parameters and diameter $d \ge 3$ is Q-polynomial, the conditions $a_d = 0$ and $\alpha > 0$ are equivalent to Γ being tight, cf. A. Pascasio [23, Theorem 1.3]. This way, we can avoid the calculation in the ($\Longleftarrow$) part.

(ii) When restricting to non-bipartite distance-regular graphs with classical parameters, the property of being tight can be seen as a generalization of being antipodal.

(iii) A distance-regular graph with classical parameters with diameter $d \ge 3$ and $b \ge 2$ is tight precisely when the parameter β meets the lower bound from Proposition 1(iii).

6 Properties of tight graphs with classical parameters

Let Γ be a distance-regular graph with diameter $d \ge 2$ and u, v two of its vertices at distance h. The local graph $\Gamma(u)$ is partitioned into three parts corresponding to the distance from v (which can only be $h-1$, h, or $h+1$). The sizes of these sets are c_h, a_h, and b_h, see Figure 1. Let us call this partition $\mathrm{CAB}_h(u,v)$. Suppose that for such partitions, we have the following regularity property: for $w \in \Gamma(u)$, the triple intersection numbers $\left[\begin{smallmatrix} u & v & w \\ 1 & i & 1 \end{smallmatrix}\right]$ only depend on i, h and $\partial(v,w)$, and not on the choice of vertices $u, v, w \in V\Gamma$. Then we say that Γ has the *CAB property*.

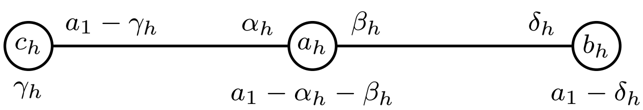

Fig. 1: The CAB_h ($1 \le h \le d-1$) partition of a local graph of a distance-regular graph with diameter d. The CAB_d partition generally consists of two parts, but for tight graphs it is trivial as $a_d = 0$ holds.

Similarly, a graph Γ is said to be 1-*homogeneous* in the sense of Nomura [21] if its triple intersection numbers $\left[\begin{smallmatrix} u & v & w \\ i & 1 & j \end{smallmatrix}\right]$ only depend on i, j, h and $\partial(v,w)$ when $\partial(u,w) = 1$, and not on the choice of vertices $u, v, w \in V\Gamma$ (note that the triangle inequality implies $|i-j| \le 1$).

Jurišić and Koolen [13, Theorem 3.1] showed that the CAB property is equivalent to 1-homogeneity. They note that a graph Γ with these properties is locally strongly regular, and also give recursive formulas to compute the triple intersection numbers in question in terms of the intersection arrays of Γ and its local graph. Suppose $w_i \in \Gamma(u)$ and $\partial(v,w_i) = i$ ($|h-i| \le 1$). Set

$$\alpha_h = \begin{bmatrix} u & v & w_h \\ 1 & h-1 & 1 \end{bmatrix}, \quad \beta_h = \begin{bmatrix} u & v & w_h \\ 1 & h+1 & 1 \end{bmatrix}, \quad \gamma_h = \begin{bmatrix} u & v & w_{h-1} \\ 1 & h-1 & 1 \end{bmatrix}, \quad \delta_h = \begin{bmatrix} u & v & w_{h+1} \\ 1 & h & 1 \end{bmatrix}.$$

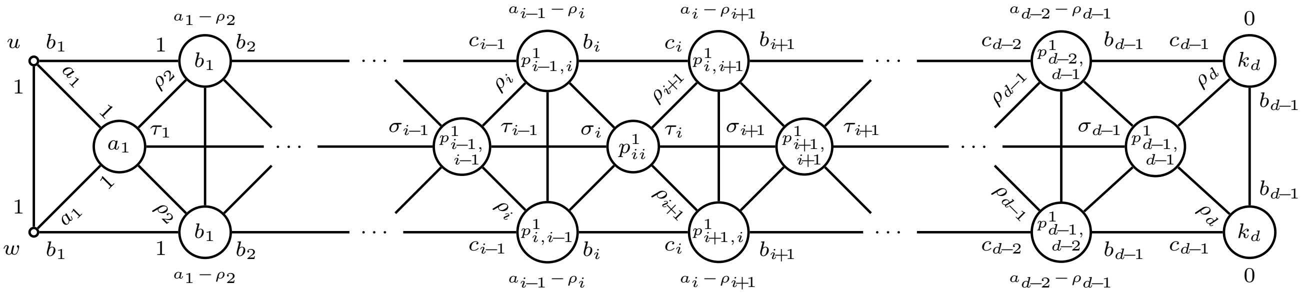

Fig. 2: The partition of a tight distance-regular graph with diameter d according to distances from two adjacent vertices. Note that the neighbourhood of u in the above partition is split into the $\mathrm{CAB}_1(u, w)$ partition, which is the distance partition of the local graph $\Gamma(u)$ corresponding to w.

The above triple intersection numbers are parameters of the $\mathrm{CAB}_h(u, v)$ partition; the remaining triple intersection numbers for the CAB property can be easily computed from these and the intersection numbers of Γ, see Figure 1. Similarly, set

$$\rho_h = \begin{bmatrix} u & v & w_{h-1} \\ h-1 & 1 & h-1 \end{bmatrix}, \quad \sigma_h = \begin{bmatrix} u & v & w_h \\ h-1 & 1 & h-1 \end{bmatrix}, \quad \tau_h = \begin{bmatrix} u & v & w_h \\ h+1 & 1 & h+1 \end{bmatrix}.$$

Again, the above triple intersection numbers determine all the remaining triple intersection numbers for the 1-homogeneous property, see also Figure 2.

Jurišić, Koolen and Terwilliger [15, Proposition 6.5] showed that Γ is tight iff it is 1-homogeneous with $a_1 \neq 0$ and $a_d = 0$, which implies that the above formulas can be applied to tight distance-regular graphs with classical parameters.

Theorem 7 *Let Γ be a tight distance-regular graph with classical parameters (d, b, α, β). Then its local graphs are strongly regular with $\mu = \alpha(b+1)$, $\lambda = (\alpha-1)(b+1)+\alpha b[d-2]_b$, eigenvalues $a_1 = \alpha(b+1)[d-1]_b$, $\theta^+ = \alpha b[d-2]_b$ and $\theta^- = -1-b$, and multiplicities*

$$1, \quad m^+ = \frac{b(b+1)[d-1]_b\,(1+\alpha[d-1]_b)}{b+1+\alpha b[d-2]_b}, \quad m^- = a_1 - m^+ - 1.$$

Moreover, Γ has the CAB and 1-homogeneous properties, and the corresponding triple intersection numbers are determined by the seven sequences $\{\alpha_h\}_{h=1}^{d}$, $\{\beta_h\}_{h=1}^{d-1}$, $\{\gamma_h\}_{h=1}^{d}$, $\{\delta_h\}_{h=1}^{d-1}$, $\{\rho_h\}_{h=2}^{d}$, $\{\sigma_h\}_{h=2}^{d-1}$ and $\{\tau_h\}_{h=1}^{d-1}$ satisfying the following closed formulas:

$$\alpha_h = 1+\alpha[h-1]_b, \quad \beta_h = b\left(1+\alpha b^h[d-h-1]_b\right), \quad \gamma_h = \alpha(b+1)[h-1]_b, \quad \delta_h = \alpha(b+1)[h]_b,$$

$$\rho_h = \alpha b^{h-2}(b+1)[h-1]_b, \; \sigma_h = [h-1]_b\,(1+\alpha[h-1]_b)\,, \; \tau_h = b^{h+1}[d-h-1]_b\left(1+\alpha b^h[d-h-1]_b\right).$$

Proof By Theorem 6, the local graph $\Gamma(u)$ for $u \in V\Gamma$ is strongly regular with valency a_1 on k vertices. Using Lemma 1, we compute its spectrum and μ, λ.

As Γ is 1-homogeneous and thus also has the CAB property, it suffices to check that the given formulas for the seven sequences of triple intersection numbers satisfy the recursive relations and initial conditions of [13, Theorem 2.4 and Remark 3.2]. □

Remark. It turns out that the graph Γ from the above result is always locally pseudo-geometric with parameters (R, K, T) where $R = b+1$, $T = \alpha$ and $K = \beta$ (for the definition see Bose [1], cf. [3, p. 440]).

We are now ready to give a classification of antipodal distance-regular graphs with classical parameters. When $d \geq 3$, these graphs have $b = 1$ by [3, Proposition 6.3.1], so we could derive the classification from [3, Theorem 6.1.1], which is based mainly on a result by Terwilliger [3, Theorem 4.4.11]. Instead, we give a new proof based on the above results on tight distance-regular graphs with classical parameters.

Corollary 3 *Let Γ be a distance-regular graph with classical parameters (d, b, α, β) and $d \geq 3$. Then Γ is antipodal if and only if it is one of the following:*

(*a*) *the d-cube* $(b = 1, \alpha = 0, \beta = 1)$,

(*b*) *the Johnson graph* $J(2d, d)$ $(b = 1, \alpha = 1, \beta = d)$,
(*c*) *the halved* $2d$*-cube* $(b = 1, \alpha = 2, \beta = 2d - 1)$,
(*d*) *the Gosset graph* $E_7(1)$ $(b = 1, \alpha = 4, \beta = 9)$.

Given a graph Γ and two vertices $u, v \in V\Gamma$ with $\partial(u, v) = 2$, we define the *μ-graph* $\Gamma(u, v)$ to be the graph induced by the common neighbours of the vertices u and v in the graph Γ.

Proof Let Γ be antipodal. By Proposition 2(ii), the classical parameters of Γ must satisfy $b = 1$, so we have $[i]_b = i$ for all i. If Γ is bipartite, we have $\alpha = 0$ and $\beta = 1$ by [3, Proposition 6.3.1(i)], which gives us precisely the d-cubes, cf. Egawa [8]. Suppose now that Γ is not bipartite, so $\alpha \neq 0$. We can easily see from (3) that α must be positive. Since Γ is tight by Proposition 2(ii), it is also 1-homogeneous by Theorem 7. We may choose vertices u, v, w of Γ such that $\partial(u, v) = \partial(v, w) = 1$ and $\partial(u, w) = 2$. The vertices u and w have $c_2 = 2(\alpha + 1)$ common neighbours. We see that $\rho_2 = 2\alpha$ of them are also neighbours of v. Therefore, there is a single common neighbour of u and w that is at distance 2 from v. The μ-graph $\Gamma(u, w)$ is therefore the Cocktail party graph $K_{(\alpha+1)\times 2}$. 1-homogeneous graphs with Cocktail party μ-graphs have been classified by Jurišić and Koolen [14]. Among them, precisely the cases (b), (c) and (d) given above have classical parameters. The converse is obvious as in all four cases we have $b = 1$ and the relation $\beta = 1 + \alpha(d - 1)$ is satisfied. □

7 A feasible 2-parameter family for primitive tight graphs

We now assume that Γ is a tight distance-regular graph with classical parameters and recall that Γ is primitive iff $b \neq 1$. Note that the only known primitive tight graph is the Patterson graph (it is related to the Suzuki group and it does not have classical parameters). We assume additionally that $\alpha = b - 1$, and obtain a two-parameter family of feasible intersection arrays, i.e., $(d, b, b - 1, b^{d-1})$. Its smallest primitive example, obtained by setting $d = 4, b = 2$, with intersection array $\{120, 98, 60, 8; 1, 6, 28, 120\}$ and 6561 vertices, is listed by Brouwer [2] in the extended tables of feasible intersection arrays. The following theorem establishes that we must have $b = 1$.

Theorem 8 *Let Γ be a distance-regular graph with diameter $d \geq 3$. Then Γ has classical parameters $(d, b, b - 1, b^{d-1})$ if and only if Γ is the d-cube $(b = 1)$. In particular, there is no distance-regular graph with intersection array $\{120, 98, 60, 8; 1, 6, 28, 120\}$.*

Proof If $b = 1$, then Γ is the d-cube, cf. Brouwer et al. [3, Proposition 1.13.1]. We will show that no examples exist when $b \neq 1$. In the case $d = 3$, we have $\beta = b^2$, which is below the lower bound from Theorem 3 when $b < 0$. Therefore, we have $b \geq 2$ and $p^3_{33} = -(b + 1)(b^2 - 1) < 0$, a contradiction. If $d \geq 4$, then we have $a_1 = (b + 1)(b^{d-1} - 1) > 0$ and $c_2 = b(b + 1) > 1$, so $b \geq 2$ must hold by Theorem 2. Now, Proposition 2(i) establishes that Γ is tight, so we can apply Theorem 7 to compute the multiplicity m^+ of the local graph of Γ:

$$\begin{aligned} m^+ &= \frac{b(b+1)[d-1]_b\,(1 + (b-1)[d-1]_b)}{b + 1 + b(b-1)[d-2]_b} = \frac{b^d(b+1)[d-1]_b}{b^{d-1}+1} \\ &= \frac{b^d(b^{d-1} + 2\sum_{i=1}^{d-2} b^i + 1)}{b^{d-1}+1} = b^d\left(1 + \frac{2b[d-2]_b}{b^{d-1}+1}\right). \end{aligned}$$

The denominator of the fractional part in the above equation is coprime to b and b^d, and $0 < 2[d-2]_b < b^{d-1} + 1$ then implies that m^+ is nonintegral. Therefore, the local graph of Γ, and consequently also Γ, do not exist. The converse is obvious. □

Jack Koolen suggested that we generalize our approach. Let Γ be a tight distance-regular graph with classical parameters (d, b, α, β) and $\Delta := \Gamma(u)$ its local graph at some vertex u of Γ. Since the smallest eigenvalue of Δ is $\theta^- = -1 - b$ by Theorem 7, it follows by Neumaier [20] that for any given positive integer b, there is only a finite number of possible local graphs distinct from $K_{m\times n}$ and the block graph of $\mathrm{OA}(m, n)$ or $S(2, m, n)$. In the first case, by [3, Proposition 1.1.5], the graph Γ is the complete multipartite graph $K_{(n+1)\times m}$ with $n \geq 2$ and $d = 2$. In the remaining two cases, Theorem 7 implies $\alpha = b = m - 1$ and $n = [d]_b$, or $\alpha = b + 1 = m$ and $n = (b + 1)[d]_b$. If Γ has diameter 3, the intersection number $p^3_{33} = \alpha b(1 - b^2)/(1 + \alpha)$ is nonnegative only when $b = 1$, in which case the graph Γ must then be either the Johnson graph $J(6, 3)$, the halved 6-cube, or the Gosset graph $E_7(1)$ by Corollary 3. We are thus interested in the case when $d \geq 4$ and $b \geq 2$, and $\alpha = b$ or $\alpha = b + 1$ depending on Δ

being the block graph of an orthogonal array or a Steiner system. We use Lemma 2 to compute the multiplicity m_d for $\alpha = b$ and $\alpha = b+1$:

$$\prod_{i=2}^{d} \left(1 + b^{i-1} \frac{b^d - 1}{b^i - 1}\right) \quad \text{and} \quad \frac{\prod_{i=1}^{d-1} \left(1+b^{d-i}(b+1)[i-1]_b\right) \left(b^i+[d+i]_b+[d+i-1]_b\right)}{(b+1)[d-1]_b \prod_{i=2}^{d-1} \left(1+b^i\left([d-i]_b+[d-i-1]_b\right)\right)\left([d-i+1]_b+[d-i]_b\right)},$$

respectively. There are no known integral solutions for $b \geq 2$. Using a computer, we have verified that for any choice of $b \geq 2$, integrality of m_d implies $d > 17$ in the first case and $d > 7$ in the second case. These calculations give us confidence to posit the following conjecture.

Conjecture 2 Let Γ be a tight distance-regular graph with classical parameters (d, b, α, β), $b \geq 2$ and diameter $d \geq 4$. For $u \in V\Gamma$, the local graph $\Gamma(u)$ is not the block graph of an orthogonal array or a Steiner system.

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