The Q-polynomial idempotents of a distance-regular graph

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May 9, 2010

Abstract

We obtain the following characterization of Q-polynomial distance-regular graphs. Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a minimal idempotent of Γ which is not the trivial idempotent E_0. Let $\{\theta^*_i\}_{i=0}^d$ denote the dual eigenvalue sequence for E. We show that E is Q-polynomial if and only if (i) the entry-wise product $E \circ E$ is a linear combination of E_0, E, and at most one other minimal idempotent of Γ; (ii) there exists a complex scalar β such that $\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}$ is independent of i for $1 \leq i \leq d-1$; (iii) $\theta^*_i \neq \theta^*_0$ for $1 \leq i \leq d$.

1 Introduction

In this paper we give a new characterization of the Q-polynomial property for distance-regular graphs. In order to motivate and describe our result, we first recall some notions. Let Γ denote a distance-regular graph with diameter $d \geq 3$ and vertex set X (see Section 2 for formal definitions). Recall that there exists a minimal idempotent E_0 of Γ such that $E_0 = |X|^{-1}J$, where J is the all 1's matrix. We call E_0 *trivial*. Let $\{E_i\}_{i=1}^d$ denote an ordering of the nontrivial minimal idempotents of Γ. It is known that for $0 \leq i, j \leq d$ the entry-wise product $E_i \circ E_j$ is a linear combination of the minimal idempotents of Γ, so that

$$E_i \circ E_j \;=\; \frac{1}{|X|} \sum_{h=0}^{d} q^h_{ij}\, E_h.$$

The coefficients q^h_{ij} are called the *Krein parameters* of Γ. They are real and nonnegative; see for example [2, p. 48–49]. Now consider when is a Krein parameter zero. Note that $J \circ E_j = E_j$ for $0 \leq j \leq d$, so $q^h_{0j} = \delta_{hj}$ for $0 \leq h, j \leq d$. The ordering $\{E_i\}_{i=1}^d$ is called *Q-polynomial* whenever for all $0 \leq i, j \leq d$ the Krein parameter q^1_{ij} is zero if $|i-j| > 1$ and nonzero if $|i-j| = 1$. Let E denote a nontrivial minimial idempotent of Γ. We say that E is *Q-polynomial* whenever there exists a Q-polynomial ordering $\{E_i\}_{i=1}^d$ of the nontrivial minimal idempotents of Γ such that $E = E_1$.

We now explain the Q-polynomial property in terms of representation diagrams. Again, let $\{E_i\}_{i=1}^d$ denote an ordering of the nontrivial minimal idempotents of Γ, and abbreviate $E = E_1$.

The *representation diagram* Δ_E is the undirected graph with vertex set $\{0, \ldots, d\}$ such that vertices i, j are adjacent whenever $i \neq j$ and $q^1_{ij} \neq 0$. By our earlier comments, $q^1_{0j} = \delta_{1j}$ for $0 \leq j \leq d$. Therefore, in Δ_E the vertex 0 is adjacent to the vertex 1 and no other vertex, see Figure 1(a). Observe that E is Q-polynomial if and only if Δ_E is a path, and in this case the natural ordering $0, 1, \ldots$ of the vertices in Δ_E agrees with the Q-polynomial ordering associated with E. See Figure 1(c).

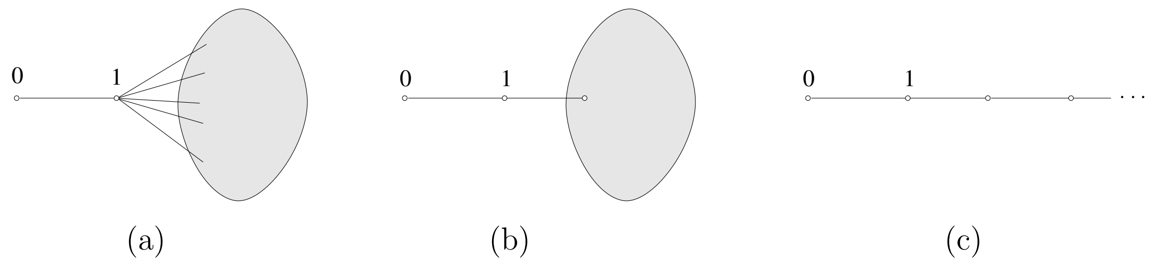

Figure 1: For $E = E_1$ the representation diagram Δ_E around vertices 0 and 1: (a) in general; (b) when $E = E_1$ is a tail; (c) when E is Q-polynomial.

Let E denote a nontrivial minimal idempotent of Γ. In this paper we give a condition that is necessary and sufficient for E to be Q-polynomial. We now describe the condition, which has three parts.

The first part has to do with the representation diagram Δ_E. According to Lang [4], E is a *tail* whenever $E \circ E$ is a linear combination of E_0, E, and at most one other minimal idempotent of Γ. In terms of the diagram Δ_E, and writing $E = E_1$ for notational convenience, E is a tail if and only if vertex 1 is adjacent to at most one vertex besides vertex 0, see Figure 1(b). Note that if E is Q-polynomial, then E is a tail.

The next part of our condition involves the dual eigenvalue sequence $\{\theta^*_i\}^d_{i=0}$ for E. This sequence satisfies $E = |X|^{-1} \sum_{i=0}^d \theta^*_i A_i$, where $\{A_i\}^d_{i=0}$ are the distance matrices of Γ. Following [4], we say that E is *three-term recurrent* (in short TTR) whenever there exists a complex scalar β such that $\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}$ is independent of i for $1 \leq i \leq d-1$. If E is Q-polynomial, then E is TTR by [2, Theorem 8.1.2], cf. [6].

The third part of our condition involves both the diagram Δ_E and the dual eigenvalues $\{\theta^*_i\}^d_{i=0}$. By [2, Proposition 2.11.1], Δ_E is connected if and only if $\theta^*_i \neq \theta^*_0$ for $1 \leq i \leq d$. These equivalent statements hold if E is Q-polynomial, since in this case Δ_E is a path.

We now state our main result.

Theorem 1.1. *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial minimal idempotent for Γ and let $\{\theta^*_i\}^d_{i=0}$ denote the corresponding dual eigenvalue sequence. Then E is Q-polynomial if and only if*

(i) *E is a tail,*

(ii) *E is TTR,*

(iii) *$\theta^*_i \neq \theta^*_0$ for $1 \leq i \leq d$.*

In Section 4 we discuss the minimality of assumptions (i)–(iii) of Theorem 1.1. We show that in general, no proper subset of (i)–(iii) is sufficient to imply that E is Q-polynomial. Theorem 1.1 gives a characterization of the Q-polynomial distance-regular graphs. A similar characterization, where assumption (i) is replaced by some equations involving the dual eigenvalues and intersection numbers, is given by Pascasio [7].

2 Preliminaries

In this section we review some definitions and basic concepts. See Brouwer, Cohen and Neumaier [2] and Terwilliger [8] for more background information. Let $\mathbb{C}$ denote the complex number field and X a nonempty finite set. Let $\mathrm{Mat}_X\mathbb{C}$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. Observe that $\mathrm{Mat}_X\mathbb{C}$ acts on V by left multiplication. For all $y \in X$, let $\hat{y}$ denote the element of V with 1 in the y-th coordinate and 0 in all other coordinates.

From now on Γ denotes a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, the shortest path-length distance function ∂ and diameter $d := \max\{\partial(x,y) \,|\, x, y \in X\}$. For a vertex $x \in X$ and integer $i \geq 0$ define

$$\Gamma_i(x) = \{y \in X \,|\, \partial(x,y) = i\}.$$

For notational convenience abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$, the graph Γ is said to be *regular with valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$. The graph Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq d)$ and vertices $x, y \in X$ with $\partial(x,y) = h$, the number $p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of x, y. The constants p^h_{ij} are called the *intersection numbers* of Γ. From now on assume Γ is distance-regular with diameter $d \geq 3$. Note that Γ is regular with valency $k = p^0_{11}$.

We recall the Bose-Mesner algebra of Γ. For $0 \leq i \leq d$ let A_i denote the matrix in $\mathrm{Mat}_X\mathbb{C}$ with (x,y)-entry equal to 1 if $\partial(x,y) = i$ and 0 otherwise. We call A_i the *i-th distance matrix* of Γ. Note that A_i is real and symmetric. We observe that $A_0 = I$, where I is the identity matrix, and abbreviate $A = A_1$. We observe that $\sum_{i=0}^d A_i = J$ and $A_iA_j = \sum_{h=0}^d p^h_{ij} A_h$ for $0 \leq i, j \leq d$. Let M denote the subalgebra of $\mathrm{Mat}_X\mathbb{C}$ generated by A. By [2, p. 127] the matrices $\{A_i\}_{i=0}^d$ form a basis for M. We call M the *Bose-Mesner algebra* of Γ. By [2, p. 45], M has a basis $\{E_i\}_{i=0}^d$ such that (i) $E_0 = |X|^{-1}J$; (ii) $I = \sum_{i=0}^d E_i$; (iii) $E_iE_j = \delta_{ij}E_i$ for $0 \leq i, j \leq d$. By [1, p. 59, 64] the matrices $\{E_i\}_{i=0}^d$ are real and symmetric. We call $\{E_i\}_{i=0}^d$ the *minimal idempotents* of Γ. We call E_0 *trivial*. For $0 \leq i \leq d$ let m_i denote the rank of E_i. Since $\{E_i\}_{i=0}^d$ form a basis for M, there exist complex scalars $\{\theta_i\}_{i=0}^d$ such that

$$A = \sum_{i=0}^d \theta_i E_i. \tag{1}$$

By (1) and since $E_iE_j = \delta_{ij}E_i$ we have

$$AE_i \;=\; E_iA \;=\; \theta_i E_i \qquad (0 \le i \le d). \tag{2}$$

We call the scalar θ_i the *eigenvalue* of Γ corresponding to E_i. Note that the eigenvalues $\{\theta_i\}_{i=0}^d$ are mutually distinct since A generates M. Moreover $\{\theta_i\}_{i=0}^d$ are real, since A and $\{E_i\}_{i=0}^d$ are real. Let E denote a minimal idempotent of Γ. Since $\{A_i\}_{i=0}^d$ form a basis for M, there exist complex scalars $\{\theta_i^*\}_{i=0}^d$ such that

$$E \;=\; \frac{1}{|X|}\sum_{i=0}^{d} \theta_i^*\, A_i. \tag{3}$$

We call θ_i^* the *i-th dual eigenvalue* of Γ corresponding to E. Note that $\{\theta_i^*\}_{i=0}^d$ are real, since E and $\{A_i\}_{i=0}^d$ are real. Let $\circ$ denote the entry-wise product in $\mathrm{Mat}_X\mathbb{C}$. Observe $A_i \circ A_j = \delta_{ij}A_i$ for $0 \le i, j \le d$, so M is closed under $\circ$. Therefore there exist complex scalars q_{ij}^h such that

$$E_i \circ E_j = \frac{1}{|X|}\sum_{h=0}^{d} q_{ij}^h E_h \qquad (0 \le i, j \le d). \tag{4}$$

We call the q_{ij}^h the *Krein parameters* of Γ. These parameters are real and nonnegative [2, p. 48–49]. For the moment fix integers h, i, j $(0 \le h, i, j \le d)$. By construction $q_{ij}^h = q_{ji}^h$. By [2, Lemma 2.3.1] we have $m_h q_{ij}^h = m_i q_{jh}^i = m_j q_{hi}^j$. Therefore

$$q_{ij}^h = 0 \quad \text{iff} \quad q_{jh}^i = 0 \quad \text{iff} \quad q_{hi}^j = 0. \tag{5}$$

We recall the dual Bose-Mesner algebras of Γ [8, p. 378]. For the rest of this section, fix a vertex $x \in X$. For $0 \le i \le d$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X\mathbb{C}$ with (y, y)-entry

$$(E_i^*)_{yy} \;=\; \begin{cases} 1; & \text{if } \partial(x, y) = i \\ 0; & \text{if } \partial(x, y) \ne i \end{cases} \qquad (y \in X).$$

We call E_i^* the *i-th dual idempotent of* Γ *with respect to* x. We observe that $I = \sum_{i=0}^d E_i^*$ and $E_i^*E_j^* = \delta_{ij}E_i^*$ for $0 \le i, j \le d$. Therefore the matrices $\{E_i^*\}_{i=0}^d$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X\mathbb{C}$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x.

For $0 \le i \le d$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X\mathbb{C}$ with (y, y)-entry

$$(A_i^*)_{yy} \;=\; |X|\,(E_i)_{xy} \qquad (y \in X).$$

We call A_i^* the *dual distance matrix* of Γ corresponding to E_i. By [8, p. 379] the matrices $\{A_i^*\}_{i=0}^d$ form a basis for M^*. Select an integer ℓ $(1 \le \ell \le d)$ and set $E := E_\ell$, $A^* := A_\ell^*$. Let $\{\theta_i^*\}_{i=0}^d$ denote the dual eigenvalues corresponding to E. Then using (3) we find

$$A^* \;=\; \sum_{i=0}^{d} \theta_i^* E_i^*. \tag{6}$$

Moreover,

$$A^*E_i^* = E_i^*A^* = \theta_i^*E_i^* \qquad (0 \le i \le d). \tag{7}$$

We now recall how M and M^* are related. By the definition of the distance matrices and dual idempotents, we have

$$E_h^*A_iE_j^* = 0 \text{ if and only if } p_{ij}^h = 0 \qquad (0 \le h, i, j \le d). \tag{8}$$

By [8, Lemma 3.2],

$$E_hA_i^*E_j = 0 \text{ if and only if } q_{ij}^h = 0 \qquad (0 \le h, i, j \le d). \tag{9}$$

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X\mathbb{C}$ generated by M and M^*. We call T the *subconstituent algebra* or *Terwilliger algebra* of Γ with respect to x [8, p. 380]. By a T*-module* we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and contains no T-modules other than 0 and W.

We mention a special irreducible T-module. Let $\boldsymbol{j} = \sum_{y \in X} \hat{y}$ denote the all 1's vector in V. Observe that $A_i\hat{x} = E_i^*\boldsymbol{j}$ and $|X|E_i\hat{x} = A_i^*\boldsymbol{j}$ for $0 \le i \le d$. Therefore $M\hat{x} = M^*\boldsymbol{j}$. Denote this common space by V_0 and note that V_0 is a T-module. This T-module is irreducible by [8, Lemma 3.6]. For $0 \le i \le d$ the vector $A_i\hat{x}$ is a basis for $E_i^*V_0$ and $E_i\hat{x}$ is a basis for E_iV_0. We call V_0 the *primary* T-module.

3 The main result

In this section we prove our characterization of Q-polynomial distance-regular graphs.

Lemma 3.1. *Let Γ denote a distance-regular graph with diameter $d \ge 3$. Let E denote a nontrivial minimal idempotent of Γ that is TTR and let $\{\theta_i^*\}_{i=0}^d$ be the corresponding dual eigenvalues. Let β and γ^* denote complex scalars such that $\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* = \gamma^*$ for $1 \le i \le d-1$. Then the expression*

$$\theta_{i-1}^{*\,2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) \tag{10}$$

is independent of i for $1 \le i \le d$.

Proof. Let δ_i^* denote the expression in (10). For $1 \le i \le d-1$ the difference $\delta_i^* - \delta_{i+1}^*$ is equal to

$$(\theta_{i-1}^* - \theta_{i+1}^*)(\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* - \gamma^*),$$

and is therefore 0. It follows that δ_i^* is independent of i for $1 \le i \le d$. □

Lemma 3.2. *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial minimal idempotent of Γ that is TTR. Fix a vertex x of Γ and let $A^* = A^*(x)$ denote the dual distance matrix corresponding to E. Then*

$$0 = [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \delta^* A], \tag{11}$$

where β and γ^* are from Lemma 3.1 and δ^* is the common value of (10). Here $[r, s]$ means $rs - sr$.

Proof. Let C denote the expression on the right in (11). We show that $C = 0$. Observe

$$C = ICI = \left(\sum_{i=0}^{d} E_i^*\right) C \left(\sum_{j=0}^{d} E_j^*\right) = \sum_{i=0}^{d}\sum_{j=0}^{d} E_i^* C E_j^*.$$

To show $C = 0$ it suffices to show $E_i^* C E_j^* = 0$ for $0 \leq i, j \leq d$. For notational convenience define a polynomial P in two variables

$$P(u, v) = u^2 - \beta uv + v^2 - \gamma^*(u + v) - \delta^*.$$

For $0 \leq i, j \leq d$ we have

$$E_i^* C E_j^* = E_i^* A E_j^* \, P(\theta_i^*, \theta_j^*) \, (\theta_i^* - \theta_j^*)$$

by (7), where $\{\theta_i^*\}_{i=0}^d$ are the dual eigenvalues corresponding to E. By (8) we find $E_i^* A E_j^* = 0$ if $|i - j| > 1$. By Lemma 3.1 we find $P(\theta_i^*, \theta_j^*) = 0$ if $|i - j| = 1$. Of course $\theta_i^* - \theta_j^* = 0$ if $i = j$. Therefore $E_i^* C E_j^* = 0$ as desired. We have now shown $C = 0$. □

In (9) we gave a characterization of a vanishing Krein parameter. We will need a variation on that result. We will obtain this variation after a few lemmas. The first lemma follows directly from the definitions of $\hat{x}$ and the dual adjacency matrices.

Lemma 3.3. *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial minimal idempotent of Γ. Fix a vertex x of Γ and let $A^* = A^*(x)$ denote the dual distance matrix corresponding to E. Then $A^* v = |X|(E\,\hat{x}) \circ v$ for all $v \in V$.* □

Lemma 3.4. *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and let $\{E_i\}_{i=0}^d$ denote the minimal idempotents of Γ. Fix $x \in X$. Then for $0 \leq h, i, j \leq d$,*

$$E_h A_i^* E_j \, \hat{x} = q_{ij}^h \, E_h \, \hat{x},$$

where $A_i^* = A_i^*(x)$.

Proof. From Lemma 3.3 we find $A_i^* E_j \, \hat{x} = |X|(E_i \, \hat{x}) \circ (E_j \, \hat{x})$, and observe that this equals $|X|(E_i \circ E_j)\, \hat{x}$. Now

$$E_h A_i^* E_j \, \hat{x} = |X| \, E_h (E_i \circ E_j) \, \hat{x} = E_h \sum_{\ell=0}^{d} q_{ij}^{\ell} E_\ell \, \hat{x} = q_{ij}^h E_h \, \hat{x}. \qquad \square$$

Corollary 3.5. *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and let $\{E_i\}_{i=0}^d$ denote the minimal idempotents of Γ. Fix $x \in X$ and let V_0 denote the primary module for $T(x)$. Then the following* (i), (ii) *are equivalent for* $0 \leq h, i, j \leq d$.

(i) $q_{ij}^h = 0$.

(ii) *$E_h A_i^* E_j$ vanishes on V_0, where $A_i^* = A_i^*(x)$.*

Suppose (i), (ii) *fail. Then $E_h A_i^* E_j V_0 = E_h V_0$.*

Proof. (i) $\Longrightarrow$ (ii) $E_h A_i^* E_j$ is zero by (9) and hence vanishes on V_0.

(ii) $\Longrightarrow$ (i) Observe $\hat{x} \in V_0$ so $E_h A_i^* E_j \hat{x} = 0$. Therefore $q_{ij}^h E_h \hat{x} = 0$ in view of Lemma 3.4. The vector $E_h \hat{x}$ is a basis for $E_h V_0$ so $E_h \hat{x} \neq 0$. Thus $q_{ij}^h = 0$.

Suppose (i), (ii) fail. Then $E_h A_i^* E_j V_0$ is a nonzero subspace of the one-dimensional space $E_h V_0$ and is therefore equal to $E_h V_0$. □

We are now ready to prove our main result.

Proof of Theorem 1.1. First suppose that E is Q-polynomial. Then condition (i) holds by definition of a tail and the definition of the Q-polynomial property, condition (ii) holds by [2, Theorem 8.1.2], cf. Leonard [6], and condition (iii) holds by [2, Proposition 4.1.8].

To obtain the converse, assume that E satisfies (i)–(iii). We show E is Q-polynomial. To do this we consider the representation diagram Δ_E from the introduction. We will show that Δ_E is a path.

Let $\{E_i\}_{i=1}^d$ denote an ordering of the nontrivial minimal idempotents of Γ such that $E = E_1$. Let X denote the vertex set of Γ. Fix $x \in X$ and let $A^* = A_1^*(x)$. We first show that Δ_E is connected. To do this we follow an argument given in [9, Theorem 3.3]. Suppose that Δ_E is not connected. Then there exists a nonempty proper subset S of $\{0, 1, \ldots, d\}$ such that i, j are not adjacent in Δ_E for all $i \in S$ and $j \in \{0, 1, \ldots, d\} \backslash S$. Invoking (9) we find $E_i A^* E_j = 0$ and $E_j A^* E_i = 0$ for $i \in S$ and $j \in \{0, 1, \ldots, d\} \backslash S$. Define $F := \sum_{i \in S} E_i$ and observe

$$A^* F = I A^* F = \left(\sum_{i=0}^{d} E_i \right) A^* F = F A^* F.$$

By a similar argument $F A^* = F A^* F$, so A^* commutes with F. Since $F \in M$ there exist complex scalars $\{\alpha_i\}_{i=0}^d$ such that $F = \sum_{i=0}^d \alpha_i A_i$. We have

$$0 = A^* F - F A^* = \sum_{i=1}^{d} \alpha_i (A^* A_i - A_i A^*). \tag{12}$$

We claim that the matrices $\{A^* A_i - A_i A^* \,|\, 1 \leq i \leq d\}$ are linearly independent. To prove the claim, for $1 \leq i \leq d$ define $B_i = A^* A_i - A_i A^*$, and observe $B_i \hat{x} = (\theta_i^* - \theta_0^*) A_i \hat{x}$. The vectors $\{A_i \hat{x}\}_{i=1}^d$ are linearly independent and $\theta_i^* \neq \theta_0^*$ for $1 \leq i \leq d$ so the vectors $\{B_i \hat{x}\}_{i=1}^d$ are linearly independent. Therefore the matrices $\{B_i\}_{i=1}^d$ are linearly independent and the claim is proved. By the claim and (12) we find $\alpha_i = 0$ for $1 \le i \le d$. Now $F = \alpha_0 I$. But $F^2 = F$ so $\alpha_0^2 = \alpha_0$. Thus $\alpha_0 = 0$, in which case $S = \emptyset$, or $\alpha_0 = 1$, in which case $S = \{0, 1, \ldots, d\}$. In either case we have a contradiction so Δ_E is connected.

As we mentioned in the introduction, in Δ_E the vertex 0 is adjacent to vertex 1 and no other vertex of Δ_E. By the definition of a tail and since Δ_E is connected, vertex 1 is adjacent to vertex 0 and exactly one other vertex in Δ_E. To show that Δ_E is a path, it suffices to show that each vertex in Δ_E is adjacent to at most two other vertices in Δ_E. We assume this is not the case and obtain a contradiction. Let v denote a vertex in Δ_E that is adjacent to more than two vertices of Δ_E. Of all such vertices, we pick v such that the distance to 0 in Δ_E is minimal. Call this distance i. Note that $2 \le i \le d-1$ by our above comments and the construction. For notational convenience and without loss of generality we may assume that the vertices of Δ_E are labelled such that for $1 \le j \le i-1$ the vertex j is adjacent to $j-1$ and $j+1$ and no other vertex in Δ_E. By construction the chosen vertex v is labelled i. This vertex is adjacent to $i-1$ and at least two other vertices in Δ_E. Let t denote a vertex in Δ_E other than $i-1$ that is adjacent to vertex i. Since E is TTR there exists $\beta \in \mathbb{C}$ such that $\theta^*_{j-1} - \beta\theta^*_j + \theta^*_{j+1}$ is independent of j for $1 \le j \le d-1$. We claim that

$$\theta_t - (\beta+1)\theta_i + (\beta+1)\theta_{i-1} - \theta_{i-2} = 0. \tag{13}$$

To prove the claim we consider the equation (11). In that equation we expand the right-hand side to get

$$0 = A^{*3}A - (\beta{+}1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3} - \gamma^*(A^{*2}A - AA^{*2}) - \delta^*(A^*A - AA^*).$$

In this equation we multiply each term on the left by E_{i-2} and on the right by E_t. To help simplify the results we make some comments. Using (2) we find $E_{i-2}A^{*3}AE_t = E_{i-2}A^{*3}E_t\,\theta_t$. Using (9) we find

$$E_{i-2}A^{*3}E_t = E_{i-2}A^*\left(\sum_{r=0}^d E_r\right)A^*\left(\sum_{s=0}^d E_s\right)A^*E_t = E_{i-2}A^*E_{i-1}A^*E_iA^*E_t.$$

Similarly we calculate

$$\begin{aligned}
E_{i-2}A^{*2}AA^*E_t &= E_{i-2}A^*E_{i-1}A^*E_iA^*E_t\,\theta_i, \\
E_{i-2}A^*AA^{*2}E_t &= E_{i-2}A^*E_{i-1}A^*E_iA^*E_t\,\theta_{i-1}, \\
E_{i-2}AA^{*3}E_t &= E_{i-2}A^*E_{i-1}A^*E_iA^*E_t\,\theta_{i-2}
\end{aligned}$$

and

$$\begin{aligned}
E_{i-2}A^{*2}AE_t &= 0, & E_{i-2}AA^{*2}E_t &= 0, \\
E_{i-2}A^*AE_t &= 0, & E_{i-2}AA^*E_t &= 0.
\end{aligned}$$

From these comments we find

$$0 = E_{i-2}A^*E_{i-1}A^*E_iA^*E_t\big(\theta_t - (\beta+1)\theta_i + (\beta+1)\theta_{i-1} - \theta_{i-2}\big). \tag{14}$$

We show that $E_{i-2}A^*E_{i-1}A^*E_iA^*E_t \neq 0$. By the last assertion of Corollary 3.5 and since the sequence $(i-2, i-1, i, t)$ is a path in Δ_E, we find $E_iA^*E_tV_0 = E_iV_0$ and $E_{i-1}A^*E_iV_0 = E_{i-1}V_0$ and $E_{i-2}A^*E_{i-1}V_0 = E_{i-2}V_0$. Therefore

$$E_{i-2}A^*E_{i-1}A^*E_iA^*E_tV_0 = E_{i-2}V_0.$$

Observe $E_{i-2}V_0 \neq 0$ so $E_{i-2}A^*E_{i-1}A^*E_iA^*E_t \neq 0$, as desired. By this and (14) we obtain (13). By (13) the scalar θ_t is uniquely determined. The scalars $\theta_0, \ldots, \theta_d$ are mutually distinct so t is uniquely determined, for a contradiction. We have shown that Δ_E is a path and therefore E is Q-polynomial. □

4 Remarks

In this section we make some remarks concerning the three conditions in Theorem 1.1. Throughout the section assume Γ is a distance-regular graph with valency k, diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Pick a nontrivial minimal idempotent $E = E_j$ of Γ and let $\{\theta^*_i\}_{i=0}^d$ denote the corresponding dual eigenvalue sequence. Abbreviate $\theta = \theta_j$.

Note 4.1. [2, pp. 142–143, 161] Pick an integer i $(1 \leq i \leq d)$. Then $\theta^*_i = \theta^*_0$ if and only if at least one of the following holds.
(i) Γ is bipartite, i is even, and $j = d$,
(ii) Γ is antipodal, $i = d$, and j is even.

Note 4.2. The graph Γ is imprimitive if and only if Γ is bipartite or antipodal [2, Theorem 4.2.1]. Thus if Γ is primitive then $\theta^*_i \neq \theta^*_0$ for $1 \leq i \leq d$.

Lemma 4.3. *Assume that $\theta \neq -1$ and one of the following occurs:*
(i) $d = 3$,
(ii) $d = 4$, Γ *is antipodal, and* j *is even,*
(iii) $d = 5$, Γ *is antipodal, and* j *is even.*

Then E is TTR.

Proof. We have $(\theta^*_1 - \theta^*_2)kp^1_{12} = \theta^*_0(k-\theta)(1+\theta)$ by [3, Lemma 2.3], so $\theta^*_1 \neq \theta^*_2$. Define $\beta \in \mathbb{C}$ such that $\beta + 1 = (\theta^*_0 - \theta^*_3)/(\theta^*_1 - \theta^*_2)$. By the construction $\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}$ is independent of i for $i = 1, 2$. We are done in case (i), so assume we are in cases (ii) or (iii). By [2, p. 142] we have $\theta^*_i = \theta^*_{d-i}$ for $0 \leq i \leq d$. Therefore $\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}$ is independent of i for $1 \leq i \leq d-1$. In other words, E is TTR. □

Note 4.4. Referring to the conditions (i)–(iii) of Theorem 1.1, we show that no proper subset of (i)–(iii) implies that E is Q-polynomial.

- Conditions (ii), (iii) are not sufficient. Assume Γ is the generalized hexagon of order $(2, 1)$ [2, p. 200, 425]. It is primitive with diameter $d = 3$ and eigenvalues $4, 1 + \sqrt{2}, 1 - \sqrt{2}, -2$. Pick $j = 1$. Then E satisfies condition (ii) by Lemma 4.3 and E satisfies condition (iii) by Note 4.2. But E does not satisfy (i) by [2, p. 413, 425]. In particular E is not Q-polynomial.
- Conditions (i), (iii) are not sufficient. Assume Γ is the dodecahedron, which is antipodal with diameter $d = 5$ and eigenvalues $3, \sqrt{5}, 1, 0, -2, -\sqrt{5}$, see [2, p. 417]. Pick $j = 1$. Then E satisfies condition (i) by [2, p. 413, 417] and E satisfies condition (iii) by Note 4.1. By [3, Lemma 2.2] we find $\theta^*_0 = 3$, $\theta^*_1 = \sqrt{5}$, $\theta^*_2 = 1$, $\theta^*_3 = -1$, $\theta^*_4 = -\sqrt{5}$, $\theta^*_5 = -3$ and using this one verifies that E does not satisfy condition (ii). In particular E is not Q-polynomial.
- Conditions (i), (ii) are not sufficient. Assume Γ is the Wells graph [2, p. 421], which is antipodal with diameter $d = 4$ and eigenvalues $5, \sqrt{5}, 1, -\sqrt{5}, -3$. Pick $j = 2$. Then E satisfies condition (i) by [2, p. 413] and E satisfies condition (ii) by Lemma 4.3, but E does not satisfy condition (iii) by Note 4.1. In particular E is not Q-polynomial.

Note 4.5. Assume Γ is bipartite, and consider the conditions (i)–(iii) of Theorem 1.1. If E satisfies conditions (i), (iii) then E is Q-polynomial by [4, proof of Theorem 5.4]. If E satisfies conditions (ii), (iii) then E is Q-polynomial by [5, Theorem 10.5].

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