On Graphs with Complete Multipartite μ-Graphs

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September 7, 2008

Abstract

Jurišić and Koolen proposed to study 1-homogeneous distance-regular graphs, whose μ-graphs (that is, the graphs induced on the common neighbours of two vertices at distance 2) are complete multipartite. Examples include the Johnson graph J$(8,4)$, the halved 8-cube, the known generalized quadrangle of order $(4,2)$, an antipodal distance-regular graph constructed by T. Meixner and the Patterson graph. We investigate a more general situation, namely, requiring the graphs to have complete multipartite μ-graphs, and that the intersection number α exists, which means that for a triple (x,y,z) of vertices in Γ, such that x and y are adjacent and z is at distance 2 from x and y, the number $\alpha(x,y,z)$ of common neighbours of x, y and z does not depend on the choice of a triple. The latter condition is satisfied by any 1-homogeneous graph. Let $K_{t\times n}$ denote the complete multipartite graph with t parts, each of which consists of an n-coclique. We show that if Γ is a graph whose μ-graphs are all isomorphic to $K_{t\times n}$ and whose intersection number α exists, then $\alpha = t$, as conjectured by Jurišić and Koolen, provided $\alpha \geq 2$. We also prove $t \leq 4$, and that equality holds only when Γ is the unique distance-regular graph $3.O_7(3)$.

Keywords: distance-regular graphs, μ-graphs, complete multipartite graphs, local graphs, generalized quadrangles, regular points

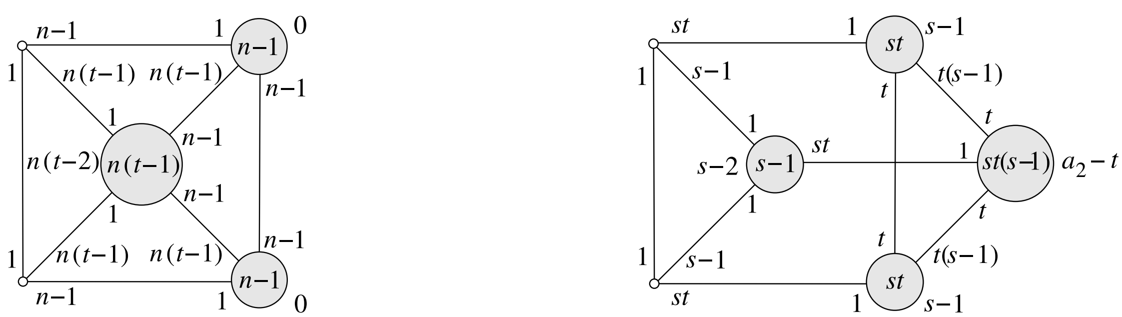
1 Introduction

Let Γ be a connected graph with a pair of nonadjacent vertices. Then the subgraph induced on the set of common neighbours of two vertices at distance 2 is called a ***μ-graph*** of Γ. Let $\boldsymbol{K_{t\times n}}$ denote the complete multipartite graph consisting of t parts of size n, i.e., the complement of t copies of K_n; see Figure 1.1(a). In this paper, we investigate graphs whose μ-graphs are isomorphic to $K_{t\times n}$, where t and n are positive integers independent of the choice of a μ-graph. When $n = 1$, such graphs are known as Terwilliger graphs and we refer the reader to [1, Sect. 1.16] for the current status of research on these graphs. The graph $K_{t\times 2}$ is called a cocktail party graph, and distance-regular graphs whose μ-graphs are $K_{t\times 2}$ have been investigated by Jurišić and Koolen [5]. They extended their work to the more general case, where μ-graphs are $K_{t\times n}$ in [6]. The main motivation came from antipodal tight distance-regular graphs of diameter 4. While not all such distance-regular graphs have complete multipartite μ-graphs, these graphs enjoy an extra property that is a consequence of the 1-homogeneous property (see Figure 1.1): we say that the intersection number **$\alpha(\Gamma)$ exists** if, for a triple of vertices (x,y,z) of Γ such that x and y are adjacent and z is at distance 2 from both x and y, the number of common neighbours of x,y,z is a constant independent of (x,y,z), and we denote this constant by $\boldsymbol{\alpha = \alpha(\Gamma)}$. To avoid the degenerate case, we assume that there exists at least one such triple (x, y, z) in Γ when we say $\alpha(\Gamma)$ exists. In the case of distance-regular graphs this assumption is equivalent to $a_2 \neq 0$. Jurišić and Koolen [6] proposed the following problem.

Problem 1 ([6]). Let Γ be a distance-regular graph with diameter at least 2, whose μ-graphs are the complete multipartite graph $K_{t\times n}$ with $n \geq 2$, and the intersection number $\alpha(\Gamma)$ exists with $\alpha(\Gamma) \geq 2$. Then show $\alpha(\Gamma) = t$.

In [5, Lemma 2.1] it was shown that $\alpha(\Gamma) \in \{t-1, t\}$. See also Lemma 7 and [5, Remark 3.6, Conjecture 3.7] for some partial information in the case when $\alpha(\Gamma) = 1$.

One of the main purposes of this paper is to provide an affirmative answer to the above problem. In fact, we do not need to assume that Γ is distance-regular. The key step in the proof is the reduction lemma, which states that by taking a local graph of Γ one obtains a graph whose μ-graphs are isomorphic to $K_{(t-1)\times n}$ and α exists, although the value of the intersection number α is smaller by 1 than that of Γ. This reduction lemma allows us to restrict the possible values of t and n, since the case $\alpha = t = 1$ corresponds to generalized quadrangles whose extensions have been investigated in detail. In particular, we show $t \leq 4$, and that equality holds only when Γ is the unique distance-regular graph $3.O_7(3)$.



(a) (b)

Figure 1.1: The distance partition corresponding to a pair of adjacent vertices in Γ. When this partition is equitable, we say that the graph is **1-homogeneous**. This means that we can write numbers beside the edges connecting any two cells that indicate how many neighbours a vertex from the closer cell has in the other cell and also put beside each cell the valency of the graph induced by its vertices. Finally, we put inside the cell the number of its vertices.
(a) the complete multipartite graph $K_{(t+1)\times n}$, with the intersection array $\{tn, n-1; 1, tn\}$,
(b) the collinearity graph of $\mathrm{GQ}(s,t)$ with the intersection array $\{s(t+1), st; 1, t+1\}$ and $a_1 = s-1$, $a_2 = (s-1)(t+1)$.

2 Convexity and generalized quadrangles

We use the standard terminology in graph theory, as in [1]. Let $\Gamma = (V, E)$ be a graph. If S is a subset of V, then by abuse of notation, we also denote by S the induced subgraph of Γ on S. A subset C of the vertices V is said to be **convex** if it contains all the shortest paths between any two of its vertices. The **convex closure** of a subset of vertices is the smallest convex subset containing it. For vertices $v_1, \ldots, v_n$ of Γ we denote by $\boldsymbol{\Gamma(v_1, \ldots, v_n)}$ the set of their common neighbours. If $S = \{v_1, \ldots, v_n\}$, then we also denote this set by $\boldsymbol{\Gamma(S)}$. For a vertex $v \in V$ we denote by $\boldsymbol{\Gamma_i(v)}$ the set of vertices at distance i from v.

Let Γ be a graph. For a vertex x of Γ we call the graph induced by $\Gamma(x)$ the **local graph** of Γ with respect to x. If X is a graph, or a family of graphs, or a property of graphs, then we say Γ is locally X if every local graph of Γ is isomorphic to X, is isomorphic to a member of X, has the property X, respectively. Recall some regularity properties of graphs.

(R1) Any two adjacent vertices have precisely λ common neighbours, i.e., every local graph is λ-regular.

(R2) Any two vertices at distance 2 have precisely μ common neighbours.

(R3) Any two nonadjacent vertices have precisely μ common neighbours.

A regular graph with v vertices and valency k is called **edge-regular** with parameters (v, k, λ) if (R1) holds, **amply regular** with parameters (v, k, λ, μ) if (R1) and (R2) hold, **co-edge-regular** with parameters (v, k, μ) if (R3) holds, and **strongly regular** with parameters (v, k, λ, μ) if (R1) and (R3) hold. For a regular graph Γ, (R1) and (R3) are complementary, in the sense that Γ satisfies (R1) if and only if the complement of Γ satisfies (R3).

We have already mentioned in the introduction that graphs with all μ-graphs isomorphic to $K_{t\times n}$ turn out to be related to generalized quadrangles. A **generalized quadrangle** $\mathrm{GQ}(s,t)$ is an incidence structure of points and lines such that

- on each line there are exactly $s+1$ points (*sitting*),
- *through* each point there are exactly $t+1$ lines, and
- for every nonincident pair (p, ℓ) of a point and a line there is exactly one line through the point p that intersects the line ℓ (or equivalently, there is exactly one point on the line ℓ that is collinear with the point p).

For a detailed treatment of generalized quadrangles see Payne and Thas [8], Hirschfeld and Thas [4]. There a distinguished property of a point being *regular* is studied. For a point p of an incidence structure of points and lines we denote by $\boldsymbol{p^\perp}$ the set of all points that are collinear with p, and for a subset A of points we denote the intersection of the sets $p^\perp$ for $p \in A$ by $\boldsymbol{A^\perp}$. To define this property in a generalized quadrangle $\mathrm{GQ}(s,t)$ we consider two noncollinear points p, q and note that

$$|\{p,q\}^\perp| = t+1 \qquad \text{and} \qquad |\{p,q\}^{\perp\perp}| \le t+1.$$

As usual, we pay a special attention to the case when an inequality is satisfied with equality. So we say that a point p is **regular** when equality holds in this inequality for every point q that is noncollinear with p. Let Γ be the collinearity graph (i.e., the point graph) of $\mathrm{GQ}(s,t)$. Then the graph Γ is strongly regular with parameters

$$v = (s+1)(st+1), \quad k = s(t+1), \quad \lambda = s-1 \quad \text{and} \quad \mu = t+1, \tag{1}$$

and $\alpha(\Gamma) = 1$; see Figure 1.1(b). The convex closure of the vertices in Γ that correspond to the points p and q, is the complete bipartite graph $K_{t+1,t+1}$. A generalized quadrangle with all points regular is called **regular**. In regular generalized quadrangles $\mathrm{GQ}(s,t)$ with $s=t$, the sets $|\{p,q\}^{\perp\perp}|$, where (p,q) is a pair of noncollinear points, have the same size as lines and can be used to construct new generalized quadrangles. The following result provides some explanation why regular generalized quadrangles motivate our study.

Lemma 1 (convexity). *For integers $t, n \ge 2$ let Γ be a connected graph of diameter at least 2, in which every μ-graph is isomorphic to $K_{t\times n}$. For vertices x, y at distance 2 in Γ, set $\boldsymbol{K(x,y)} = \Gamma(x,y) \cup \Gamma(\Gamma(x,y))$. Then the following statements hold:*

(i) *$\Gamma(\Gamma(x,y)) \cong \overline{K_n}$;*

(ii) *$K(x,y) \cong K_{(t+1)\times n}$;*

(iii) *$K(x,y)$ is the convex closure of the set $\{x,y\}$.*

Proof. Set $\Delta = \Gamma(x,y) \cong K_{t\times n}$. Since $n \geq 2$, there exist nonadjacent vertices $x', y' \in \Delta$. Setting $\Delta' = \Gamma(x',y')$, we find $\Delta \cap \Delta' \cong K_{(t-1)\times n} \neq \emptyset$ by the assumption. Set

$$C = \Delta \setminus (\Delta \cap \Delta') = \Gamma(x,y) \cap \Gamma(\Delta \cap \Delta') \cong \overline{K_n} \supseteq \{x', y'\},$$
$$C' = \Delta' \setminus (\Delta \cap \Delta') = \Gamma(x',y') \cap \Gamma(\Delta \cap \Delta') \cong \overline{K_n} \supseteq \{x, y\},$$

with other words, C (resp. C') is the maximal independent set of vertices in Δ (resp. Δ') that contains x', y' (resp. x, y). Since $t \geq 2$ there exists a pair of nonadjacent vertices $x'', y'' \in \Delta \cap \Delta'$. Then $C \cup C' \subset \Gamma(x'', y'') \cong K_{t\times n}$. Since $C \cap C' = \emptyset$, we have $C' \subset \Gamma(C)$. Thus

$$C' = C' \cap \Gamma(C) = \Gamma(x',y') \cap \Gamma(\Delta \cap \Delta') \cap \Gamma(C) = \Gamma(\{x',y'\} \cup (\Delta \cap \Delta') \cup C) = \Gamma(\Delta) = \Gamma(\Gamma(x,y)).$$

This proves the statement (i). Furthermore, $K(x,y) = \Delta \cup C' = \Delta \cup \Gamma(\Delta)$ by the above equality, which means $K(x,y) \cong K_{(t+1)\times n}$ and so we have also proved (ii).

The convex closure of $\{x,y\}$ contains Δ and thus also $\Gamma(\Delta)$, so altogether it contains $K(x,y)$. We have $K(x,y) \cong K_{(t+1)\times n}$ by (ii), the graph $K_{(t+1)\times n}$ has diameter 2 and $\mu(K_{(t+1)\times n}) = nt = \mu(\Gamma)$, so we conclude that the set $K(x,y)$ is convex. Therefore, we have shown (iii) as well. □

The statements (ii), (iii) in the above result and the following lemma were first shown in [7, Lemma 4.1] and [5, Lemma 2.1] for distance-regular graphs. We included short proofs in order to make our presentation self-contained.

Lemma 2 (bounds). *For integers $t, n \geq 2$ let Γ be a connected graph in which every μ-graph is isomorphic to $K_{t\times n}$. If the intersection number $\alpha(\Gamma)$ exists, then for any triple of vertices (x,y,z) of Γ such that $d(x,y) = 1$, $d(x,z) = d(y,z) = 2$, we have $\Gamma(x,y,z) \cong K_{\alpha(\Gamma)}$. Moreover, $\alpha(\Gamma) \in \{t-1, t\}$.*

Proof. By the definition of the intersection number $\alpha(\Gamma)$, we have $|\Gamma(x,y,z)| = \alpha(\Gamma)$. If $\Gamma(x,y,z) = \Gamma(x,z) \cap \Gamma(y)$ contains a pair of nonadjacent vertices u and v, then the vertices x, y and z all belong to $\Gamma(u,v) \cong K_{t\times n}$. Since $K_{t\times n}$ cannot contain three vertices x,y,z with $d(x,y) = 1$, $d(x,z) = d(y,z) = 2$, we obtain a contradiction. Thus $\Gamma(x,y,z)$ is a clique. Since the largest clique of $\Gamma(x,z) \cong K_{t\times n}$ has size t, we conclude $\alpha(\Gamma) \leq t$. If $\alpha(\Gamma) \leq t-2$, then there are at least two cocliques of $\Gamma(x,z) \cong K_{t\times n}$ which are adjacent to all the vertices of $\Gamma(x,y,z)$. Thus there are adjacent vertices $u, w \in \Gamma(x,z) \cap \Gamma(\Gamma(x,y,z)) \subset \Gamma(x,z) \setminus \Gamma(x,y,z) \subset \Gamma_2(y)$, so

$$\alpha(\Gamma) = |\Gamma(u,w,y)| \geq |\{x\} \cup \Gamma(x,y,z)| = 1 + \alpha(\Gamma).$$

This is a contradiction. Hence $\alpha(\Gamma) \geq t-1$. □

3 Reduction to local graphs

The complete multipartite graph $K_{(t+1)\times n}$ has the property that its μ-graphs are the complete multipartite graphs $K_{t\times n}$, however, the intersection number α is not defined. It is maybe even more important that its local graphs are of the same form as the original graph, namely they are complete multipartite graphs $K_{t\times n}$. Before we use this recursive property in our situation, we need the following result that turned out to be very important.

Lemma 3 (regularity). *Let μ, μ' be nonnegative integers satisfying $\mu > \mu' + 1$. Let Γ be a connected graph in which for every pair of vertices x_1, x_2 at distance 2, the graph $\Gamma(x_1, x_2)$ is regular of valency μ' on μ vertices. Then Γ is regular.*

Proof. Since Γ is connected, it suffices to show $|\Gamma(x)| = |\Gamma(y)|$ for every pair of adjacent vertices x, y of Γ. We do it by a two-way counting of the edges between the sets $\Gamma(x)\cap\Gamma_2(y)$ and $\Gamma(y)\cap\Gamma_2(x)$ (this also shows that one of these two sets is empty if and only if the other one is). A vertex $z \in \Gamma(x)\cap\Gamma_2(y)$ (if it exists) has one neighbour in $\Gamma(y)\cap\Gamma_0(x)$ (namely the vertex x), μ' neighbours in $\Gamma(y)\cap\Gamma_1(x)$ (by the definition of μ') and all other neighbours in $\Gamma(y)$ are (there are $\mu - \mu' - 1$ of them) in $\Gamma(y)\cap\Gamma_2(x)$ (by the triangle inequality $\Gamma(y)\cap\Gamma_i(x) = \emptyset$ for $i > 2$). Therefore, we have

$$(\mu - \mu' - 1)\,|\Gamma(x)\cap\Gamma_2(y)| = \sum_{z\in\Gamma(x)\cap\Gamma_2(y)} |\Gamma(y,z)\cap\Gamma_2(x)|$$

and by symmetry also

$$(\mu - \mu' - 1)\,|\Gamma_2(x)\cap\Gamma(y)| = \sum_{w\in\Gamma_2(x)\cap\Gamma(y)} |\Gamma(x,w)\cap\Gamma_2(y)|.$$

Since these two numbers both count the edges between the sets $\Gamma(x)\cap\Gamma_2(y)$, $\Gamma_2(x)\cap\Gamma(y)$, and $\mu \neq \mu' + 1$ by our assumption, we conclude

$$|\Gamma(x)\cap\Gamma_2(y)| = |\Gamma_2(x)\cap\Gamma(y)|,$$

which implies $|\Gamma(x)| = 1 + |\Gamma(x)\cap\Gamma(y)| + |\Gamma(x)\cap\Gamma_2(y)| = 1 + |\Gamma(x)\cap\Gamma(y)| + |\Gamma_2(x)\cap\Gamma(y)| = |\Gamma(y)|$. □

A special case ($\mu' = 0$) of the above result is due to Enomoto [2]; see also [1, p.4].

Lemma 4 (reduction). *For integers $t, n \geq 2$ let Γ be a connected graph of diameter at least 2, in which every μ-graph is isomorphic to $K_{t\times n}$. Then Γ is regular. Moreover, for an arbitrary vertex x of Γ, the subgraph $\Delta = \Gamma(x)$ satisfies the following properties:*

(i) *Δ is regular;*

(ii) *Δ has diameter 2 and every μ-graph of Δ is isomorphic to $K_{(t-1)\times n}$;*

(iii) *Δ is strongly regular if $t \geq 3$;*

(iv) *if the intersection number $\alpha(\Gamma)$ exists, then $\alpha(\Gamma) > 0$ and the intersection number $\alpha(\Delta)$ exists with $\alpha(\Delta) = \alpha(\Gamma) - 1$.*

Proof. We have $\mu = tn$, the valency of the μ-graph is $\mu' = (t-1)n$ and $\mu - \mu' - 1 = n - 1 > 0$, so the graph Γ is regular by Lemma 3. Since Γ is connected, noncomplete and regular, there exists a vertex $u \in \Gamma_2(x)$, so Δ contains $\Gamma(x, u) \cong K_{t\times n}$. In particular, as $n \geq 2$, the subgraph Δ is noncomplete. Let $y, z \in \Delta$ be nonadjacent vertices. Then $\Delta(y, z) = \Gamma(x, y, z)$ is a local graph of $\Gamma(y, z) \cong K_{t\times n}$, so $\Delta(y, z) \cong K_{(t-1)\times n}$. Thus (ii) holds by $t \geq 2$. We have $\mu(\Delta) = (t-1)n$, $\mu'(\Delta) = (t-2)n$ and $\mu(\Delta) - \mu'(\Delta) - 1 = n - 1 > 0$, so we obtain (i) by Lemma 3.

If $t \geq 3$, then applying (i) to the graph Δ, we see that it is edge-regular. This, together with (i) and (ii), establishes (iii).

Finally, suppose that the intersection number $\alpha(\Gamma)$ exists. A graph which is locally a complete multipartite graph is, by [1, Proposition 1.1.5], either triangle-free or complete multipartite. Neither is possible in our situation, since we assumed $t, n \geq 2$ and that $\alpha(\Gamma)$ exists. So the valency of Δ is at most $|\Delta| - 3$ and being at distance 0 or 2 is not an equivalence relation in Δ. (i.e., Δ is not antipodal). Therefore, there exists a pair of adjacent vertices at distance 2 from a vertex in Δ (note that these distances are the same in Γ as in Δ, since the diameter of Δ is equal to 2). This means that we can pick vertices $y, z, w \in \Gamma(x)$ with $d(z, w) = 1$, $d(y, z) = d(y, w) = 2$ and that $\alpha(\Gamma)$ is at least 1, since x is their common neighbour. Then $x \in \Gamma(y, z, w) \cong K_{\alpha(\Gamma)}$ by Lemma 2, so $|\Delta(y, z, w)| = \alpha(\Gamma) - 1$. Thus (iv) holds. □

By Lemma 4, we are lead to consider the case where the value of t is the smallest, namely $t = 1$. In this case, μ-graphs of Γ are cocliques, i.e., Γ contains no $K_{1,1,2}$ as an induced subgraph, hence the following lemma applies, provided Γ contains a triangle.

Lemma 5. *Let Γ be a connected regular graph. Assume that Γ contains a triangle but no $K_{1,1,2}$ as an induced subgraph. If the intersection number $\alpha(\Gamma)$ exists, then $\alpha(\Gamma) = 1$.*

Proof. Suppose the intersection number $\alpha(\Gamma)$ exists. If $\alpha(\Gamma) \geq 2$, then for a triple (x, y, z) of vertices with $d(x, y) = 1$, $d(x, z) = d(y, z) = 2$, we have $|\Gamma(x, y, z)| \geq 2$. Let u, v be distinct vertices in $\Gamma(x, y, z)$. If the vertices u, v are adjacent, then $\{x, z, u, v\} \cong K_{1,1,2}$, which is a contradiction. If the vertices u, v are nonadjacent, then $\{x, y, u, v\} \cong K_{1,1,2}$, which is also a contradiction. Therefore $\alpha(\Gamma) \leq 1$.

Next suppose that $\alpha(\Gamma) = 0$. Since the graph Γ is not triangle-free, we can choose a triangle (w, x, y) of Γ. Then $\Gamma(w) \cap \Gamma_2(x) \cap \Gamma_2(y) = \emptyset$. Since Γ contains no induced subgraphs $K_{1,1,2}$, we have $\Gamma(w, x) \cap \Gamma_2(y) = \emptyset$ and $\Gamma(w, y) \cap \Gamma_2(x) = \emptyset$. Therefore,

$$\Gamma(w) \cup \{w\} \subseteq \{x, y\} \cup \Gamma(x, y) \subseteq \{x\} \cup \Gamma(x).$$

Since Γ is regular, equality holds above, hence $\Gamma(w) \cup \{w\} = \Gamma(x) \cup \{x\}$. Now if we take $z \in \Gamma(x) \setminus \{w\}$, then (z, w, x) is a triangle, so we can argue just as above to conclude $\Gamma(z) \cup \{z\} = \Gamma(x) \cup \{x\}$. We have shown that $\Gamma(x) \cup \{x\}$ is a clique. Since Γ is connected, this forces the graph Γ to be complete, contradicting the hypothesis that the intersection number $\alpha(\Gamma)$ exists. □

4 A solution of the problem

The following result is essentially [5, Lemma 2.1], although the assumptions here are weaker.

Lemma 6. *Let Γ be an amply regular graph with parameters (v, k, λ, μ), which is locally co-edge-regular with parameters (k, λ, μ'). If the intersection number $\alpha(\Gamma)$ exists, then*

$$\alpha(\Gamma)\, |\Gamma_2(x) \cap \Gamma(y)| = \mu\, (\lambda - \mu')$$

whenever x, y are vertices at distance 2.

Proof. The result follows by a two-way counting of the edges between the sets $\Gamma(x,y)$ and $\Gamma(y) \cap \Gamma_2(x)$:

$$\sum_{z \in \Gamma_2(x) \cap \Gamma(y)} |\Gamma(x,y,z)| = \sum_{w \in \Gamma(x,y)} |\Gamma_2(x) \cap \Gamma(y,w)| = \sum_{w \in \Gamma(x,y)} (|\Gamma(y,w)| - |\Gamma(x,y,w)|).$$

□

Lemma 7. *For an integer $n \geq 2$ let Γ be a graph in which every μ-graph is isomorphic to $K_{n,n}$. If the intersection number $\alpha(\Gamma)$ exists and $\alpha(\Gamma) = 1$, then Γ has diameter at least 3.*

Proof. Suppose the intersection number $\alpha(\Gamma)$ exists and $\alpha(\Gamma) = 1$. Hence the diameter of Γ is at least 2. Let Δ be a local graph of Γ. Then, by Lemma 4, the graph Δ is regular, it has diameter 2, every μ-graph in Δ is isomorphic to $\overline{K_n}$, and the intersection number $\alpha(\Delta)$ exists with $\alpha(\Delta) = 0$. In particular, Δ is co-edge-regular, and Δ contains no $K_{1,1,2}$ as an induced subgraph. By Lemma 5, Δ is triangle-free, so Δ is edge-regular, hence strongly regular with parameters $(v', k', 0, n)$, where $v' = 1 + k' + k'(k'-1)/n$. By Lemma 4, Γ is regular. Since Δ is regular, Γ is edge-regular. Now suppose that Γ has diameter 2. Then Γ is strongly regular with parameters (v, k, λ, μ), where $k = v' = 1 + k' + k'(k'-1)/n$, $\lambda = k'$, and $\mu = 2n$. By Lemma 6 and $\mu' = n$, we have

$$1 + k' + \frac{k'(k'-1)}{n} - 2n = 2n(k' - n).$$

Therefore,

$$\lambda = 2n^2 - 2n + 1 \quad \text{and} \quad k = 2n(2n^2 - 3n + 2).$$

Then, by [1, p. 8], the nontrivial eigenvalues of Γ are: $r = 2n^2 - 2n$, $s = -2n + 1$ and the multiplicity f of the eigenvalue r is

$$f = \frac{(s+1)k(k-s)}{\mu(s-r)} = 8n^4 - 32n^3 + 66n^2 - 86n + 80 - \frac{120n - 84}{2n^2 - 1}.$$

However, $(120n - 84)/(2n^2 - 1) = 12(10n - 7)/(2n^2 - 1)$ is not an integer for any integer $n \geq 2$ (for note $\gcd(2n^2 - 1, 12) = 1$ and $2n^2 - 1 > 10n - 7$ for $n \geq 5$). This contradiction proves that Γ has diameter at least 3. □

Theorem 8. *For integers $t, n \geq 2$ let Γ be a connected graph in which every μ-graph is isomorphic to $K_{t \times n}$. If the intersection number $\alpha(\Gamma)$ exists with $\alpha(\Gamma) \geq 2$, then $\alpha(\Gamma) = t$.*

Proof. Let us assume the intersection number $\alpha(\Gamma)$ exists. Then there exist vertices x, y, z of Γ such that $d(x,y) = 1$, $d(x,z) = d(y,z) = 2$. By Lemma 2, we have $\alpha(\Gamma) \in \{t-1, t\}$. Suppose $\alpha(\Gamma) = t - 1$. Then $t \geq 3$ by $\alpha(\Gamma) \geq 2$. Taking local graphs successively and using Lemma 4, we obtain a graph Δ of diameter 2, in which every μ-graph is isomorphic to $K_{n,n}$ and $\alpha(\Delta) = 1$. This contradicts Lemma 7. □

Theorem 8 gives an affirmative answer to Problem 1.

5 Extensions of generalized quadrangles

Let Γ be a graph. A clique C of Γ is called **regular** if every vertex outside C is adjacent to the same number $e > 0$ of vertices in C. We call e the **nexus** of C.

Lemma 9. *Let Γ be a connected regular graph. Assume that Γ contains a triangle but no $K_{1,1,2}$ as an induced subgraph, and that the intersection number $\alpha(\Gamma)$ exists. If Γ is co-edge-regular with parameters (v, k, μ), then one of the following statements holds:*

(i) *for every vertex x, the local graph $\Gamma(x)$ consists of μ cliques of size k/μ, each of which is a regular clique with nexus 1, when x is adjoined. In particular, Γ is strongly regular with parameters $\left(v, k, \dfrac{k}{\mu} - 1, \mu\right)$; or*

(ii) *$\mu = 2$, and for every vertex x, the local graph $\Gamma(x)$ consists of two cliques of size s_+ and s_-, each of which is a regular clique with nexus 1, when x is adjoined, where*

$$s_\pm = \frac{k \pm \sqrt{k^2 - 4(v - k - 1)}}{2}.$$

Proof. Suppose the graph Γ is co-edge-regular with parameters (v, k, μ). Lemma 5 implies $\alpha(\Gamma) = 1$. Since Γ contains no induced subgraphs $K_{1,1,2}$, every edge is contained in a unique maximal clique. Thus Γ is locally a disjoint union of cliques. Let x be a vertex of Γ, M a maximal clique of Γ containing x. We claim

$$|M \cap \Gamma(z)| = 1 \quad \text{for all } z \in \Gamma_2(x). \tag{2}$$

Clearly $|M \cap \Gamma(z)| \leq 1$ since otherwise $\Gamma(x, z)$ would have an edge, producing $K_{1,1,2}$ as an induced subgraph. If $M \cap \Gamma(z) = \emptyset$, then pick $w \in M$. Since Γ has diameter 2, we have $z \in \Gamma_2(w)$, hence $|\Gamma(x, w, z)| = \alpha(\Gamma) = 1$. The unique vertex u of $\Gamma(x, w, z)$ belongs to $\{x, w\} \cup \Gamma(x, w) = M$, and so $u \in M \cap \Gamma(z)$, which contradicts $M \cap \Gamma(z) = \emptyset$. Therefore, the claim is proved.

Next, let M be a maximal clique of Γ containing x. We claim

$$M \cap \Gamma(z) = \{x\} \quad \text{for all } z \in \Gamma(x) \setminus M. \tag{3}$$

Since M is a maximal clique and $z \notin M$, there exists $y \in M$ such that y and z are nonadjacent. If z is adjacent to some vertex $w \in M \setminus \{x\}$, then $\{x, y, z, w\} \cong K_{1,1,2}$, which is a contradiction. This proves (3). By (2) and (3), M is a regular clique with nexus 1.

For $w \in M \setminus \{x\}$ we have $|\Gamma(w) \cap \Gamma_2(x)| = |\Gamma(w)| - |M \setminus \{w\}| = k - (|M| - 1)$ and, by (2),

$$v - k - 1 = \sum_{z \in \Gamma_2(x)} |M \cap \Gamma(z)| = \sum_{w \in M} |\Gamma(w) \cap \Gamma_2(x)| = (|M| - 1)\big(k - (|M| - 1)\big).$$

Thus, $s = |M| - 1$ satisfies the quadratic equation $s^2 - ks + v - k - 1 = 0$. Let $s_\pm$ denote the two solutions of this equation. Then, by $|M| - 1 \in \{s_+, s_-\}$ and $v - k - 1 > 0$, we have

$$s_+ + s_- = k \quad \text{and} \quad s_\pm > 0. \tag{4}$$

Let $m_\pm$ be the number of maximal cliques of size $s_\pm + 1$ containing x. Then obviously we have

$$m_+ s_+ + m_- s_- = k.$$

This, together with (4) implies that either $m_+ m_- = 0$ or $m_+ = m_- = 1$. In the former case, all maximal cliques containing x have the same size $s + 1 = 1 + k/\mu$, because $\mu = m_+ + m_-$ by (2). In particular, Γ is strongly regular with the desired parameters. This gives (i). The latter case leads to (ii). □

Lemma 10. *Let $t \geq 2$, $n \geq 2$ be integers, and assume $(t, n) \neq (2, 2)$. Let Γ be a connected k-regular graph on v vertices in which every μ-graph is isomorphic to $K_{t\times n}$. Assume that the intersection number $\alpha(\Gamma)$ exists. Then Γ is amply regular with parameters (v, k, λ, nt), where*

$$k = \frac{(\lambda + n)\big((n-1)\lambda + n(t-1)\big)}{n^2(t-1)} \tag{5}$$

and

$$\big(\lambda - n(t-1)\big)\big(\lambda - n(nt - n - 1)\big) \geq 0. \tag{6}$$

Equality holds in (6) if and only if Γ has diameter 2.

Proof. Let $\Delta = \Gamma(x)$ be the local graph of Γ with respect to a vertex x. Observe that, by Lemma 4, Δ is co-edge-regular with parameters (k, λ_x, μ') for some integer λ_x, where $\mu' = n(t-1)$. We claim that Δ is strongly regular with parameters $(k, \lambda_x, \lambda'_x, \mu')$ for some integer λ'_x. Indeed, if Δ is triangle-free, then the claim holds with $\lambda'_x = 0$. If $t \geq 3$, then the claim follows from Lemma 4(iii). Now suppose that Δ contains a triangle and $t = 2$. In this case, Lemma 4(ii) implies that Δ contains no $K_{1,1,2}$ as an induced subgraph, and Lemma 4(iv) implies that the intersection number $\alpha(\Delta)$ exists. As $\mu' = n \geq 3$, Lemma 9 implies that Δ is strongly regular. Therefore, we have proved the claim.

Since every μ-graph of Δ is isomorphic to $K_{(t-1)\times n}$, the graph Δ is locally co-edge-regular with parameters $(\lambda_x, \lambda'_x, \mu'')$, where $\mu'' = n(t-2)$. By Lemma 4(iv), we have $\alpha(\Delta) = \alpha(\Gamma) - 1$. If $\alpha(\Gamma) = 1$, then $t = 2$ by Lemma 2. Then Lemma 7 implies that Δ has diameter at least 3, while Lemma 4(ii) implies Δ has diameter 2. This contradiction shows that $\alpha(\Gamma) \geq 2$. Now by Theorem 8, we have $\alpha(\Gamma) = t$. Therefore, Lemma 6 implies $(t-1)(\lambda_x - \mu') = \mu'(\lambda'_x - \mu'')$. Thus

$$\lambda'_x = \frac{\lambda_x}{n} + n(t-2) - t + 1. \tag{7}$$

Since $\mu'(k - 1 - \lambda_x) = \lambda_x(\lambda_x - 1 - \lambda'_x)$, we have

$$k = 1 + \frac{\lambda_x}{\mu'}(\lambda_x - 1 - \lambda'_x + \mu').$$

Substituting (7) into the above equality, we see that λ_x is a solution to (5) regarded as an equation in λ. Since (5) is a quadratic equation in λ with exactly one nonnegative solution, we conclude that λ_x is independent of the choice of x, and so we denote it by λ for the rest of the proof.

Now, for $y \in \Gamma_2(x)$, we have, by Lemma 6,

$$\begin{aligned} 0 \;\leq\; & |\Gamma(x) \setminus \big(\Gamma(y) \cup \Gamma_2(y)\big)| = k - \mu - |\Gamma_2(y) \cap \Gamma(x)| = k - \mu - \frac{\mu(\lambda - \mu')}{\alpha(\Gamma)} \\ =\; & \frac{n-1}{n^2(t-1)}\big(\lambda - n(t-1)\big)\big(\lambda - n(nt - n - 1)\big). \end{aligned} \tag{8}$$

Equality holds in (8) if and only if $\Gamma(x) \subset \Gamma(y) \cup \Gamma_2(y)$. Since x and y are arbitrary vertices at distance 2, equality holds if and only if Γ has diameter 2. □

Theorem 11. *For an integer $n \geq 3$ let Γ be a connected graph in which every μ-graph is isomorphic to $K_{n,n}$. If the intersection number $\alpha(\Gamma)$ exists and $\alpha(\Gamma) = 2$, then Γ is locally $\mathrm{GQ}(\lambda/n, n-1)$. In particular, Γ has diameter 2 if and only if Γ is locally $\mathrm{GQ}(n-1, n-1)$.*

Proof. Suppose the intersection number $\alpha(\Gamma)$ exists and $\alpha(\Gamma) = 2$. Let $\Delta = \Gamma(x)$ be the local graph of Γ with respect to a vertex x. Then by Lemma 4, Γ is regular, and by Lemma 4(i), Δ is also a regular graph in which every μ-graph is isomorphic to $\overline{K_n}$. In particular, Γ is amply regular with parameters $(v, k, \lambda, 2n)$, and Δ contains no induced subgraphs $K_{1,1,2}$. By Lemma 4(iv), the intersection number $\alpha(\Delta)$ exists with $\alpha(\Delta) = 1$. This means that Δ is not triangle-free. Therefore, we can apply Lemma 9 to Δ. As $\mu(\Delta) = n \geq 3$, the graph Δ is locally $nK_{\lambda/n}$ and every maximal clique is a regular clique with nexus 1. Hence, Δ is the collinearity graph of a generalized quadrangle $\mathrm{GQ}(\lambda/n, n-1)$, where $\lambda/n \neq 1$. By Lemma 10, Γ has diameter 2 if and only if

$$(\lambda - n)\big(\lambda - n(n-1)\big) = 0,$$

which in turn is equivalent to $\lambda = n(n-1)$. Therefore, Γ has diameter 2 if and only if Γ is locally $\mathrm{GQ}(n-1, n-1)$. □

A graph which is locally $\mathrm{GQ}(p, q)$ is called a **triangular extended generalized quadrangle** [3].

Theorem 12. *For integers $t \geq 1$ and $n \geq 3$ let Γ be a connected graph in which every μ-graph is isomorphic to $K_{t\times n}$. If the intersection number $\alpha(\Gamma)$ exists, then $t \leq 4$. Moreover, equality holds only if Γ is the unique distance-regular graph $3.O_7(3)$, which is locally locally locally $\mathrm{GQ}(2, 2)$.*

Proof. Suppose the intersection number $\alpha(\Gamma)$ exists. We may assume $t \geq 4$, as the assertion is trivial when $t \leq 3$. Let $\Gamma^{(s)}$ denote a graph obtained from Γ by taking local graphs s times. By Lemma 4, the graph $\Delta = \Gamma^{(t-2)}$ has diameter 2 and satisfies the hypotheses of Theorem 11. Thus the graph Δ is locally $\mathrm{GQ}(n-1, n-1)$. Then by [3], we have $n = 3, 4, 5, 9$ or 14. By (1), the parameters of $\Gamma^{(t-1)}$, which is the point graph of $\mathrm{GQ}(n-1, n-1)$, are:

$$v''' = n(n^2 - 2n + 2), \quad k''' = (n-1)n, \quad \lambda''' = n - 2, \quad \mu''' = n \geq 3.$$

Since Δ is strongly regular, its parameters are:

$$\begin{aligned}
k'' &= v''' = n(n^2 - 2n + 2), \\
\lambda'' &= k''' = (n-1)n, \\
\mu'' &= 2n, \\
v'' &= 1 + k'' + \frac{k''(k'' - 1 - \lambda'')}{\mu''} = \frac{1}{2}(n^2 - 2n + 2)(n^3 - 3n^2 + 5n - 1) + 1.
\end{aligned}$$

Since we have assumed that $t \geq 4$, the graph $\Gamma^{(t-3)}$ is strongly regular with parameters

$$k' = v'' = \frac{1}{2}(n^2 - 2n + 2)(n^3 - 3n^2 + 5n - 1) + 1,$$

$$\begin{aligned}
\lambda' &= k'' = n(n^2-2n+2),\\
\mu' &= 3n,\\
v' &= 1+k'+\frac{k'(k'-1-\lambda')}{\mu'}\\
&= \frac{1}{12}(n^9-10n^8+49n^7-150n^6+319n^5-488n^4+545n^3-420n^2+202n-12).
\end{aligned}$$

Then the nontrivial eigenvalues of the graph $\Gamma^{(t-3)}$ are:

$$\frac{\lambda'-\mu'\pm\sqrt{(\mu'-\lambda')^2-4(\mu'-k')}}{2} = \frac{1}{2}\Big(n^3-2n^2-n\pm(n-1)\sqrt{n(n^3-9n+12)}\Big).$$

Since $\lambda'\neq\mu'-1$, $\Gamma^{(t-3)}$ is not a conference graph, $n(n^3-9n+12)$ is a perfect square (see, [1, p. 8]). For $n\in\{3,4,5,9,14\}$, this is the case only when $n=3$.

Finally we show that t cannot be larger than 4 even in the case $n=3$. If $t>4$, then the strongly regular graph $\Gamma^{(t-4)}$ is locally $\Gamma^{(t-3)}$, and has parameters:

$$k=v'=117,\ \lambda=k'=36,\ \mu=12,\ v=1+k+\frac{k(k-1-\lambda)}{\mu}=898.$$

Its eigenvalues are 117, $12\pm\sqrt{249}$. Since this is not a conference graph, we must have integral eigenvalues. This is a contradiction.

Now, when $t=4$, we have shown that Γ is locally locally locally GQ(2, 2). Such a graph is known to be the unique distance-regular graph $3.O_7(3)$ (see [7]). □

Acknowledgement.

The authors would like to thank an anonymous referee for correcting several errors in an earlier version of the manuscript.

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