CLASSIFICATION OF THE FAMILY AT4(qs, q, q) OF ANTIPODAL TIGHT GRAPHS

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Abstract

An antipodal tight distance-regular graph with diameter four and parameters (p, q, r) is an antipodal distance-regular graph with eigenvalues $\theta_0 = q(pq + p + q)$, $\theta_1 = pq + p + q$, $\theta_2 = p$, $\theta_3 = -q$, $\theta_4 = -q^2$ and the antipodal class size r. We denote such a graph by AT4(p, q, r). It is known that an AT4(p, q, r) is 1-homogeneous (in the sense of Nomura). As a consequence, the local graphs are strongly regular, with p and $-q$ as the nontrivial eigenvalues. Jurišić conjectured that the family of antipodal tight graphs AT4(p, q, r) is finite and that, aside from the Conway-Smith graph, the Soicher2 graph, the $3.Fi_{24}^-$ graph, all graphs in this family have parameters belonging to one of the following four subfamilies:

(i) $q \,|\, p$, $r = q$, (ii) $q \,|\, p$, $r = 2$, (iii) $p = q-2$, $r = q-1$, (iv) $p = q-2$, $r = 2$.

In this paper we settle the first subfamily, that is, we show that for an AT4(qs, q, q) there are exactly 5 possibilities for the pair (s, q) and for each of them there exists an example, namely: the Johnson graph $J(8, 4)$ for $(1, 2)$, the halved 8-cube for $(2, 2)$, the $3.O_6^-(3)$ graph for $(1, 3)$, the Meixner2 graph for $(2, 4)$ and the $3.O_7(3)$ graph for $(3, 3)$. Our main tool is using the fact that the μ-graphs of the graphs in this subfamily are complete multipartite.

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1 Introduction

A big project of classifying distance-regular graphs divides them into *primitive* (i.e., the ones with all the distance graphs connected) and *imprimitive* ones, see Brouwer and Van Bon [1] or the main reference book on the subject by Brouwer, Cohen and Neumaier [3]. The latter graphs are by Smith [22] either *bipartite* and/or *antipodal* (i.e., being at maximal distance is a transitive relation). Let Γ be an antipodal distance-regular graph with diameter d. In the case of

- $d = 3$, or
- $d = 4$ and Γ bipartite

(when the antipodal quotient is complete or complete bipartite graph) many of such infinite families are known and they are related (sometimes through some strong characterizations) to important algebraic and/or combinatorial objects such as

- **finite geometries** (projective and affine geometry, generalized quadrangles,...), together with their substructures (such as ovoids or spreads),
- **codes** (e.g. Preparata, Kerdock, Reed-Muller,...), together with their properties (perfect,...),
- **groups** (e.g. Conway, McLaughlin, Higman-Sims, Ree, symplectic, Coxeter,...), and
- Hadamard matrices, Moore graphs,...

In this way one can use the theory of distance-regular graphs (in particular, various combinatorial and graph theoretic techniques, the spectral theory, the theory of orthogonal polynomials and the theory of association schemes) to study these objects and unify some diverse areas of discrete mathematics (for example various bounds and studies of their extremal objects). However, in the case

- $d = 4$ and Γ nonbipartite

only 14 examples of such graphs are known. The situation seems to get even worse for larger d

dimater	3	4	5	6	7	8	...
nonbipartite	∞	14	4	5	1	1	...
bipartite	∞	∞	6	1	4	2	...

and ends (asymptotically) with only a couple of graphs for a fixed d that are all related to two infinite families, namely the Hamming graphs and the Johnson graphs, see [3, Ch. 14].

Question. *Have we not discovered enough constructions of infinite families of such graphs yet or are there none left to be discovered?*

The first answer in this direction came from Van Bon and Brouwer [1], that antipodal distance-regular graphs cannot have classical distance-regular graphs as their antipodal quotients, with exception of the ones that are already known and the case of diameter 4, when the antipodal quotients are Hermitean forms graphs of diameter 2, see [3, pp. 285,289]. Next, Terwilliger [23] showed that all Q-polynomial antipodal distance-regular graphs of diameter at least five are already known. These results mount pressure on the case $d = 4$.

Let Γ be an antipodal distance-regular graph with diameter $d = 4$ and eigenvalues $\theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$. Then θ_0, θ_2 and θ_4 are the eigenvalues of the antipodal quotient of Γ (so it is strongly regular). In the case of a strongly regular graph, Cameron, Goethals and Seidel [5] showed that vanishing of the Krein parameter q_{11}^1 or q_{22}^2 implies very strong structural properties of the graph. For example, the local graphs of the graph and its complement are both strongly regular as well. In the case of Γ, vanishing of Krein parameters was investigated in [14] and only Krein parameters q_{11}^4 and q_{44}^4 turned out to be important. The latter one corresponds to a Krein parameter of the antipodal quotient (since super and sub scripts are all even), while vanishing of the former one is equivalent to each of the following statements:

(a) Γ is tight (in the sense of Jurišić, Koolen and Terwilliger),

(b) Γ is 1-homogeneous (in the sense of Nomura),

(c) Γ is locally strongly regular with nontrivial eigenvalues θ_2 and θ_3,

see [14], [12] and [13]. Therefore, vanishing of q_{11}^4 again implies very strong additional combinatorial structure. Only 4 of the above mentioned 14 examples of Γ turn out not to be tight.

Suppose additionally that Γ is tight. Let us denote the nontrivial eigenvalues of the local graph of Γ by p and $-q$, hence $p = \theta_2$ and $q = -\theta_3$. Furthermore, let r be the size of antipodal classes of Γ. Then we can calculate all the intersection numbers of Γ in terms of p, q and r. We denote Γ by **AT4(p, q, r)**. In [14, Section 6] a classification of the AT4 family was proposed and many feasibility conditions were derived:

$$p \geq 1,\ q \geq 2, \quad pq(p+q)/r \text{ is even}, \quad r \mid p+q, \quad r(p+1) \leq q(p+q), \quad p \geq q-2, \quad \ldots \tag{1}$$

The first three cases of equality (together with $r = p + q$) were successfully classified and considerable information was derived also in the fourth case that was studied in [10]. There the first author conjectured that the family of antipodal tight graphs AT4(p, q, r) is finite and that, aside from the Conway-Smith graph, the Soicher2 graph and the 3.Fi$_{24}^-$ graph, all graphs in this family have parameters belonging to one of the following four subfamilies:

(i) $q \mid p$, $r = q$, (ii) $q \mid p$, $r = 2$, (iii) $p = q-2$, $r = q-1$, (iv) $p = q-2$, $r = 2$.

The local graphs of the first two subfamilies are pseudogeometric and for the last two subfamilies an additional Krein parameter, namely q_{44}^4, is vanishing, which implies that for every vertex of Γ its second subconstituent is again an antipodal distance-regular graph of diameter 4, see [10]. The Soicher1 graph AT4$(2, 4, 3)$ is an example of the latter [20] and this is how one can relate one more graph out of those 14 known examples[1] to the AT4 family.

Let the ***μ*-graph** of two vertices of Γ at distance two be a graph that is induced by their common neighbours. The μ-graphs of AT4(p, q, r) are regular with valency p, since the local graph of the μ-graph is the μ-graph of the local graph, see [11, Theorem 3.1].

[1] The three graphs left out are: (a) the Wells graph, cf. [10, Remark 3.5(b)], covering the folded 5-cube, i.e., the Hermitean forms graph on $\mathbb{F}_4^2$, (b) the 3.Sym(6).2 graph covering the complement of the triangular graph $T(6)$ (the 3.Sym(6).2 graph and AT4$(1, 2, 3)$ can be defined in terms of each other, see [3, 13.2.B]), and (c) the coset graph of the shortened ternary Golay code covering the Hermitean forms graph on $\mathbb{F}_9^2$.

In [16] we proved the following characterization of the first subfamily with their μ-graphs.

Theorem 1.1 *Let Γ be an antipodal tight distance-regular graph* $\mathrm{AT4}(p,q,r)$*. If $p = 1$ (i.e., $p + q = r$) then $q = 2$, $r = 3$ and Γ is the Conway-Smith graph. If $p > 1$ then its μ-graphs are complete multipartite if and only if there exists an integer s such that $(p,q,r) = (qs,q,q)$. Each μ-graph of* $\mathrm{AT4}(sq,q,q)$ *is equal to $K_{(s+1)\times q}$.* ∎

In this paper we will classify the family $\mathrm{AT4}(qs,q,q)$. It is the one that contains the most examples out of the above four families. The special case $q = 2$ has already been considered in [14], cf. [24]. We will show that there are exactly 5 possible intersection arrays and for each of them there exists a distance-regular graph with that intersection array. More precisely, we will show the following result.

Theorem 1.2 *Let Γ be an antipodal tight graph* $\mathrm{AT4}(p,q,r)$*. Then all its μ-graphs are complete multipartite if and only if (p,q,r) is one of the following*

- *(i)* $(1,2,3)$ *and Γ is the Conway-Smith graph,* [3, p.399]
- *(ii)* $(2,2,2)$ *and Γ is the Johnson graph $J(8,4)$,* [3, p.255]
- *(iii)* $(4,2,2)$ *and Γ is the halved 8-cube,* [3, p.264]
- *(iv)* $(3,3,3)$ *and Γ is the $3.O_6^-(3)$ graph,* [3, p.399]
- *(v)* $(8,4,4)$, cf. [3, 12.4A]
- *(vi)* $(9,3,3)$. cf. [3, p.400]

Remark 1.3 In [17] we prove that the graphs $\mathrm{AT4}(8,4,4)$ and $\mathrm{AT4}(9,3,3)$ in (v) and (vi) are unique, namely the Meixner2 graph and the $3.O_7(3)$ graph respectively. The proof goes by showing that the local graph and the local graph of the local graph and so on are unique.

In order to show the main result, we will first study graphs whose μ-graphs are complete multipartite. Furthermore, our study will relate to extended geometries, for example to graphs that are locally generalized quadrangles. As a consequence of the above result we derive new existence conditions for graphs of the AT4 family, which rule out several feasible parameter sets from the table in [3, pp. 421–425] and some infinite families that were previously considered feasible.

2 Preliminaries

Let Γ be a graph with diameter d. When being at the distance d or 0 is an equivalence relation, we say that Γ is **antipodal**. For vertices $x_1,\dots,x_n$ of Γ we denote by $\boldsymbol{\Gamma(x_1,\dots,x_n)}$ the set of their common neighbours and by $\boldsymbol{\Delta(x_1,\dots,x_n)}$ the graph induced by this set. In particular, for a vertex x of Γ we call $\Delta(x)$ the **local graph** of x. The graph Γ is said to be **locally** $\mathcal{C}$, where $\mathcal{C}$ is a graph or a class of graphs, when all its local graphs are isomorphic to (respectively are members of) $\mathcal{C}$. For example, the icosahedron is locally a pentagon, and the point graphs of generalized quadrangles are locally a union of cliques.

We define $\Gamma_i(x)$ to be the set of vertices at distance i from x. For $y \in \Gamma_i(x)$ and integers j and h we denote the set $\Gamma_j(x) \cap \Gamma_h(y)$ by $D^h_j(x,y)$ and its cardinality by $p^i_{jh}(x,y)$. We say that the intersection number p^i_{jh} **exists** if $p^i_{jh}(x,y) = p^i_{jh}$ for all pairs of vertices x and y at distance i, i.e., if it is independent of choice of x and y at distance i. We denote the intersection numbers p^i_{ii}, $p^i_{i,i+1}$, $p^i_{i,i-1}$ and p^0_{ii} respectively by a_i, b_i, c_i and k_i, for $i = 0, 1, \ldots, d$. The **distance-regular graphs** are characterized as the graphs for which the set of parameters $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$, called the **intersection array** of Γ, exist, or equivalently when for all i, j and h the numbers p^i_{jh} exist. Note that a distance-regular graph is k-regular, where $k = k_1 = b_0$, and $k = a_i + b_i + c_i$. All the local graphs of Γ have k vertices and are a_1-regular. A graph Γ is **strongly regular** when it is regular and when its intersection numbers a_1 and c_2 exist. Suppose Γ is strongly regular, it has v vertices, valency k, $\lambda = a_1$ and $\mu = c_2$, then we call (v, k, λ, μ) the parameters of Γ. Since a two-way counting of edges between the neighbours and the nonneighbours of a vertex in Γ renders $k(k-\lambda-1) = (v-k-1)\mu$, we sometimes list only three parameters, for example (k, λ, μ). Note that the connected strongly regular graphs are precisely the distance-regular graphs with diameter 2. For a detailed treatment of distance-regular graphs and all the terms which are not defined here see Brouwer et al. [3] or Godsil [7] and Godsil [8].

Let Γ be a graph. As usually, we denote the distance between vertices x and y of Γ by $\partial(x, y)$. If x, y and z are vertices of Γ such that $\partial(x, y) = 1$, $\partial(x, z) = \partial(y, z) = 2$, then we define the (triple) intersection number $\alpha(x, y, z) = |\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)|$. We say that the intersection number α of Γ **exists** when $\alpha = \alpha(x, y, z)$ for all triples of vertices (x, y, z) of Γ such that $\partial(x, y) = 1$, $\partial(x, z) = \partial(y, z) = 2$.

In this paper we classify the family AT4(qs, q, q), $s, q \in \mathbb{N}$, and for that we need the following information about it, see [14, 5.2–6.4].

Proposition 2.1 *Let Γ be an antipodal tight graph* AT4(qs, q, q)*, where $s, q \in \mathbb{N}$. Then $q \neq 1$ and the intersection number α exists with $\alpha = s + 1$ and*

(i) $(sq + s + 1)q > qs > -q > -q^2$ are its nontrivial eigenvalues and its intersection array equals

$$\big\{q^2(sq+s+1), (q^2-1)(qs+1), (r-1)q(s+1), 1; 1, q(s+1), (q^2-1)(qs+1), q^2(q+s+1)\big\},$$

(ii) the local graphs of Γ are strongly regular with nontrivial eigenvalues qs, $-q$ and parameters

$$(k', \lambda', \mu') = \big(sq(q+1), q(2s-1), qs\big). \quad \blacksquare$$

Let us recall two more definitions and one result [15, Lemma 2.1]. Let $t, n \in \mathbb{N}$. We denote the complement of t cliques of size n, i.e., the complete multipartite graph $K_{n_1, n_2, \ldots, n_t}$ with $n_1 = n_2 = \ldots = n_t = n$, by $K_{t \times n}$. If a graph Γ on v vertices is regular with valency k and any two vertices of Γ at distance 2 have precisely $\mu = \mu(\Gamma)$ common neighbours, then the graph is called **co-edge-regular** with parameters (v, k, μ), see [3, p. 3]. When we know that the μ-graphs of a distance-regular graph are complete multipartite and the intersection number α exists, we can derive the following properties of local graphs and the intersection number α.

Proposition 2.2 *Let $t, n \in \mathbb{N}$ and let Γ be a distance-regular graph with diameter at least 2, whose μ-graphs are the complete multipartite graph $K_{t\times n}$, for which $a_2 \neq 0$ and the intersection number α exists with $\alpha \geq 1$. Then the following (i)–(iii) hold.*

(i) $c_2 = nt$, *for each vertex x of Γ the local graph $\Delta(x)$ is co-edge-regular with parameters (v', k', μ'), where $v' = k$, $k' = a_1$, and $\mu' = n(t-1)$. Moreover, $\alpha a_2 = c_2(a_1 - \mu')$.*

(ii) Let x and y be vertices of Γ at distance 2. Then for all $z \in D_2^1(x, y)$ the subgraph $\Delta(x, y, z)$ is complete.

(iii) $\alpha \in \{t-1, t\}$. ∎

A **generalized quadrangle GQ(s, t)** is an incidence structure of points and lines with $s+1$ points on each line, $t+1$ lines through each point, and for any point p not on a line ℓ there is exactly one point of ℓ collinear with p. If we do not require the constant line size, then we will denote the corresponding incidence structure by GQ$(*, t)$. For a detailed treatment of generalized quadrangles see [19] and [6].

3 Locally strongly regular graphs

Let Γ be a distance-regular graph with diameter $d \geq 2$ that is locally strongly regular with parameters (v', k', λ', μ') and whose μ-graphs are the complete multipartite graph $K_{t\times n}$, $n \geq 1$, $t \geq 2$. Moreover, let us assume $a_2 \neq 0$ and the intersection number α exists. We already know some properties of the local graphs of Γ.

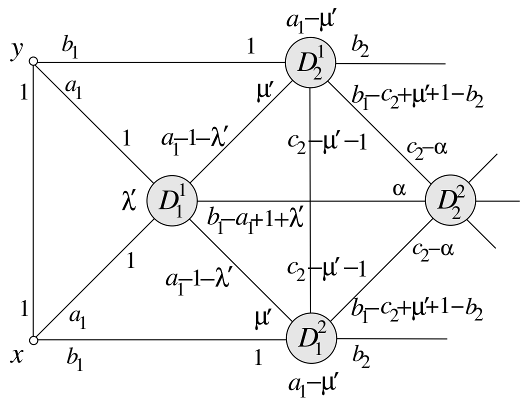

Figure 1: The distance partition corresponding to an edge xy of Γ. The number beside edges connecting cells $D_i^j(x, y)$, indicates how many neighbours a vertex from the closer cell has in the other cell. We also put beside each cell the valency of the graph induced by the vertices of it. For convenience we mention here the intersection numbers needed for the above partition: $|D_1^1| = p_{11}^1 = a_1$, $|D_1^2| = |D_2^1| = p_{12}^1 = b_1$, $|D_3^2| = |D_2^3| = p_{23}^1 = b_1 b_2/c_2$, $|D_2^2| = p_{22}^1 = a_2 b_1/c_2$.

In this section we will study the local graphs of the local graphs of Γ (in Figure 1 the graph induced by D_1^1 is such a graph). Let Δ be a subgraph of Γ. We define $\kappa(\Delta)$ to be the minimal valency of Δ. Let x, y and z be three pairwise adjacent vertices of Γ. If $d \geq 3$, $n \geq 2$ and $t \geq 4$, then we obtain the lower bound on $\kappa(\Delta(x, y, z))$, which turns out to be the best possible in certain cases as we will see in Section 5.

Theorem 3.1 *Let Γ be a distance-regular graph with diameter $d \geq 2$ that is locally strongly regular with parameters (v', k', λ', μ'). Let $t, n \in \mathbb{N}$ be such that $t \geq 2$ and let each μ-graph of Γ be the complete multipartite graph $K_{t\times n}$. Let x and y be adjacent vertices of Γ.*

(i) The subgraph $\Delta(x, y)$ is co-edge-regular with parameters (v'', k'', μ''), where $v'' = k'$, $k'' = \lambda'$ and $\mu'' = (t-2)n$, in the case when $t \geq 3$ the subgraph $\Delta(x, y)$ has diameter 2.

(ii) Let $t \geq 3$ and $z \in \Gamma(x, y)$. Then $(v'' - 1 - k'')\mu'' \leq k''\big(k'' - 1 - \kappa\big(\Delta(x, y, z)\big)\big)$, i.e.,

$$k' \leq 1 + \lambda' + \frac{\lambda'\big(\lambda' - \kappa\big(\Delta(x, y, z)\big) - 1\big)}{(t-2)n}.$$

The above inequality holds with equality if and only if for all edges xy the graph $\Delta(x, y)$ is strongly regular with parameters $(v'', k'', \lambda'', \mu'')$, where $\lambda'' = \kappa\big(\Delta(x, y, z)\big)$.

(iii) Let $a_2 \neq 0$, the intersection number α exists, $n \geq 2$, $t \geq 4$ and $d \geq 3$. Then

$$\kappa\big(\Delta(x, y, z)\big) \geq \alpha - 2 + (n-1)\big((t-3)n - (\alpha - 3)\big) \qquad \textit{for } z \in \Gamma(x, y).$$

Proof. (i),(ii) The first claim is obvious, while the second one is a straightforward consequence of the definition of $\kappa(\Delta(x, y, z))$ and the two way counting of edges between the neighbours and the nonneighbours of a vertex in the graph $\Delta(x, y)$ (that has diameter 2 by assumption $t \geq 3$).

(iii) The graph $\Delta(x, y)$ is not complete by (i). This means that there exists a vertex $u \in U := \Gamma(x, y) \cap \Gamma_2(z)$.

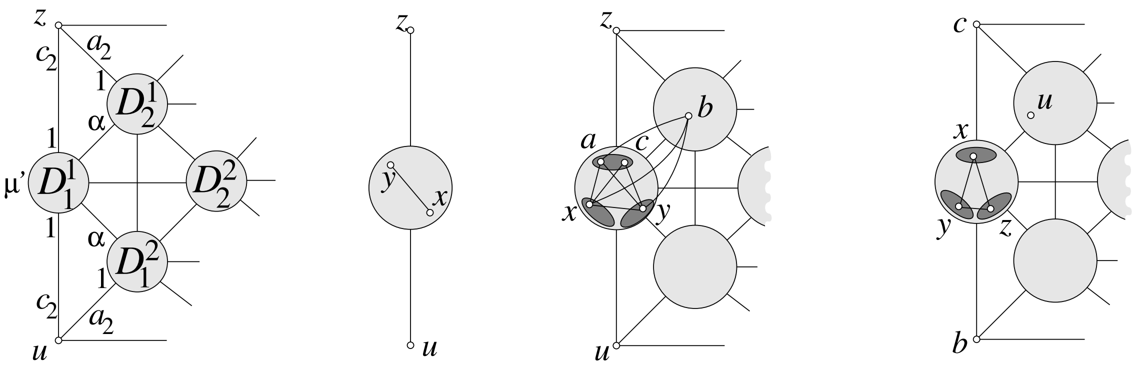

Figure 2: Parts of the distance distribution diagram of a pair of vertices at distance two (for the complete figure in the case of diameter 4 see [10, Figure 6.1]).

Vertices of the graph $\Delta(x, y, z)$ are partitioned in the following sets:

$$A := \Gamma(x, y, z, u) \qquad \text{and} \qquad B := \Gamma_2(u) \cap \Gamma(x, y, z),$$

Let us now prove the following lower bound on the valency of a vertex $b \in B$ in $\Delta(x, y, z)$:

$$|\Gamma(x, y, z, b)| \geq \alpha - 2 + (n-1)\big((t-3)n - (\alpha - 3)\big). \tag{2}$$

In the second part of the proof we will only use this bound in the situations when B is nonempty, so we do not need to consider the case $B = \emptyset$. By Proposition 2.2(ii), the vertex b has exactly $\alpha - 2$ neighbours in A, see Figure 2(c). Since $\alpha \geq 3$ by Proposition 2.2(iii) and $t \geq 4$, there exists an element $a \in A$ adjacent to b. Let C be a maximal independent set in A containing a and $c \in C \backslash \{a\}$. Note that there is $n - 1 > 0$ choices for c. By Proposition 2.2(ii), vertices a and c have no common neighbours in B, so $\partial(b, c) = 2$. Since the μ-graph of b and c is $K_{t\times n}$ and it contains x, y and z, the number of common neighbours of b and c in $\Delta(x, y, z)$ is exactly $(t-3)n$. By Proposition 2.2(ii) and $u \in D_1^2(c, b)$, exactly $\alpha - 3$ of those neighbours are adjacent to u, hence $|B \cap \Gamma(b, c)| = (t-3)n - (\alpha - 3)$. As we have already mentioned, the vertices in $C\backslash\{a\}$ have no common neighbours in B and $|C\backslash\{a\}| = n - 1$, so it follows that the vertex b has at least $(n-1)\big((t-3)n - (\alpha-3)\big)$ neighbours in B, and (2) follows.

In order to finish the proof it is enough to show that for each vertex $w \in \Gamma(x, y, z)$ there is such a vertex $u' \in \Gamma(x, y) \cap \Gamma_2(z) \cap \Gamma_2(w)$, i.e., vertices x, y, z and u' induce $K_{2,1,1}$ and u' is at distance 2 from z and w, and then apply (2) by w and u' replacing b and u. Suppose the opposite, that is that there exists a vertex $w \in \Gamma(x, y, z)$ that is adjacent to all vertices in $\Gamma(x, y) \cap \Gamma_2(z)$, i.e., $U \subseteq \Gamma(w)$. Since $u \in U$, the set U is obviously nonempty. We first look for a good lower bound on $|U|$. Let $u' \in U$ and let us denote the μ-graph of vertices u' and z in $\Delta(x, y)$ by A'. Since w is adjacent to all vertices in U, we have $w \in A'$. Since $|A'| = \mu'' = n(t-2)$ by (i), the vertex u' has $k'' - \mu''$ neighbours in U and there exists a vertex $w' \in A'\backslash\{w\}$ not adjacent to w by $n \geq 2$. In the μ-graph of w and w' the vertices z and u' are in the same coclique and all its vertices are adjacent to x and y. Thus u' is contained in a coclique of size $n-1$ in U, which implies $|U| \geq k'' - \mu'' + n - 1$. Since $A' \subseteq \Gamma(x, y, z)$ for each vertex $u' \in U$, we have

$$k'' = \mathrm{val}_{\Delta(x,y)}(w) \geq |\{z\}| + |\Gamma(w) \cap A' \cap \Gamma(x, y)| + |U| \geq 1 + (\mu'' - n) + (k'' - \mu'' + n - 1) = k''. \quad (3)$$

Therefore, $|U| = k'' - \mu'' + n - 1$, the vertex w has exactly $\mu'' - n$ neighbours in $\Gamma(x, y, z)$, i.e., the neighbours of w in $\Gamma(x, y, z)$ coincide with its neighbours in A' for each $u' \in U$. The latter implies that the neighbours of w in $\Gamma(x, y, z)$ are adjacent to all the vertices in U. Since $t \geq 4$, i.e., $\mu'' - n = (t-3)n \geq n \geq 2$, we can find two nonadjacent vertices v and v' in $\Gamma(x, y, z, w)$. Then $\{z\} \cup U \subseteq \Gamma(x, y, v, v')$. By the fact that the μ-graphs of Γ are the complete multipartite graph $K_{t\times n}$, the set U is a coclique of size $n-1$ and therefore $\mu'' = k''$. Hence, the graph $\Delta(x, y)$ is complete multipartite, and so also the graphs $\Delta(x)$ and Γ are complete multipartite by [3, Proposition 1.1.5]. But this is not possible, since we assumed $d \geq 3$. ∎

4 Locally generalized quadrangle

There are very interesting examples of the AT4(p, q, r) family that are locally generalized quadrangle (GQ). However, this is also possible in the case when a tight graph is primitive, for example the Patterson graph is locally GQ$(9, 3)$. Then there are further intriguing members of AT4 family whose local graphs are locally GQ (we will say in short **locally² GQ**), and even those that are locally³ GQ, see Table 4.1. We will investigate this phenomenon in this section.

#	graph	k	p	q	r	α	c_2	a_1	λ'	μ-graph	locally	locally^2	locally^3
$A2$	! $J(8,4)$	16	2	2	2	2	4	6	2	$K_{2,2}$	$GQ(3,1)$		
$A4$	! $3.O_6^-(3)$	45	3	3	3	2	6	12	3	$K_{3,3}$	$GQ(4,2)$		
	! Patterson	280	8	4	–	2	8	36	8	$K_{4,4}$	$GQ(9,3)$		
$A3$	! halved 8-cube	28	4	2	2	3	6	12	6	$K_{3\times 2}$	$T(8)$	$GQ(5,1)$	
$A8$	! Meixner2 [18]	176	8	4	4	3	12	40	12	$K_{3\times 4}$	$\mathcal{U}(5,2^2)$	$GQ(3,3)$	
$A6$	! $3.O_7(3)$	117	9	3	3	4	12	36	15	$K_{4\times 3}$	$O_6^+(3)$	$O_5(3)$	$GQ(2,2)$

Table 4.1: Local information of the known examples of the AT4 family and the Patterson graph that is related to generalized quadrangles. All these distance-regular graphs are characterized by their intersection arrays (for the proof of uniqueness of the Patterson graph see [4]). Note that in the case of AT4 family $\alpha = (p+q)/r$, $c_2 = q\alpha$, $a_1 = p(q+1)$, $n' = k$, $k' = a_1$, $\lambda' = 2p - q$ and $\mu' = p$, while in the case of the Patterson graph we have $\alpha = 2$, $c_2 = q\alpha$, $a_1 = q(p+1)$, $n' = k$, $k' = a_1$, $\lambda' = p$ and $\mu' = q$. Remember that in all these cases p and $-q$ denote the eigenvalues of the local graph.

Let Γ be a graph. A **geodesic** in Γ is a path $g_0, \dots, g_t$, where $\partial(g_0, g_t) = t$. Let H be a subgraph of Γ. We say that H is **convex**, when every geodesic between its vertices lies in H. The **convex closure** of H is the smallest subgraph of Γ that contains H (note that the intersection of two convex graphs is again convex, and that Γ is convex). We consider the convex closure of two vertices at distance 2 in a member of the AT4(qs, q, q) family and find out that it is also complete multipartite.

Lemma 4.1 *Let $n, t \in \mathbb{N}\backslash\{1\}$ and let Γ be a graph with diameter $d \geq 2$, whose μ-graphs are the complete multipartite graph $K_{t\times n}$. Let x and y be vertices of Γ at distance 2. Then the minimal convex subgraph of Γ containing x and y is the complete multipartite graph $K_{(t+1)\times n}$.*

Proof. Let u and v be vertices of $\Gamma(x,y)$ at distance 2. It suffices to show that the subgraph $\Gamma(x,y) \cup \Gamma(u,v)$ is a complete multipartite graph $K_{(t+1)\times n}$, since the diameter of this graph is 2 and for any two vertices at distance 2 their μ-graph is already contained in this graph. This will follow from the fact that any element $w \in \Gamma(u,v)\backslash\Gamma(x,y)$ is adjacent to all elements of $\Gamma(x,y)$. Since the μ-graph $\Delta(u,v)$ is a complete multipartite graph $K_{t\times n}$, we have $w \in D_2^2(x,y)$ and w is adjacent to all the vertices in $\Gamma(x,y,u,v)$. Let us now choose vertices $u', v' \in \Gamma(x,y)$ at distance 2 such that u and u' are adjacent. Since w belongs to the μ-graph $\Delta(u', v')$, it is adjacent also to all the vertices of $\Gamma(x,y)$ that belong to the same independent set as u and v. ∎

Let $n \in \mathbb{N}\backslash\{1\}$ and let Γ be a distance-regular graph with diameter $d \geq 2$, for which $a_2 \neq 0$, the intersection number α exists with $\alpha = 2$, and whose μ-graphs are the complete bipartite graph $K_{n,n}$. Then, by Proposition 2.2, the local graphs are co-edge regular with parameters $(v', k', \mu') = (k, a_1, n)$. Let Δ be a co-edge regular graph with parameters (v', k', μ'). We say that a pair of vertices u and v at distance 2 in Δ is **regular** if their convex closure is the complete bipartite graph $K_{\mu',\mu'}$. A vertex x of the graph Δ is **regular** if for every vertex u at distance 2 from x the pair (x,u) is regular. A point in a generalized quadrangle is regular if and only if the corresponding vertex in its point graph is regular. Regular points are very interesting object in the study of generalized quadrangles, see Payne and Thas [19]. In the following result we will show that local graphs of Γ are related to an incidence structure $\mathrm{GQ}(*, \mu'-1)$ that we introduced at the end of Section 2. We will see that all their points are regular and that the line sizes can have only two possibilities.

Lemma 4.2 *Let $n \in \mathbb{N}\backslash\{1\}$ and let Γ be a distance-regular graph with diameter $d \geq 2$, for which $a_2 \neq 0$, the intersection number α exists with $\alpha = 2$ and whose μ-graphs are the complete bipartite graph $K_{n,n}$. Let x be a vertex of Γ and Δ its local graph. Recall that Δ is a co-edge regular graph with parameters $(v', k', \mu') = (k, a_1, n)$. Then the following (i)–(iii) holds.*

(i) The local graph Δ is the point graph of a $\mathrm{GQ}(*, \mu'-1)$ with all vertices regular.

(ii) Let c be the size of a maximal clique C in Δ. Then $v' = c(k' - c + 2)$.

(iii) If the local graph Δ is strongly regular with parameters (v', k', λ', μ'), then it is the point graph of a generalized quadrangle $\mathrm{GQ}(\lambda'+1, \mu'-1)$.

Proof. (i) By Proposition 2.2, the graph Δ is co-edge regular graph with parameters $(v', k', \mu') = (k, a_1, n)$. Let u and v be its vertices at distance 2. By Lemma 4.1, their convex closure in Γ is the complete multipartite graph $K_{3\times\mu'}$ and their convex closure in Δ is the complete bipartite graph $K_{\mu',\mu'}$. This means that the local graph Δ does not contain an induced $K_{2,1,1}$, i.e., any two adjacent vertices of Δ lie in a unique clique. Furthermore, the graph Δ is locally a disjoint union of exactly μ' cliques and for every maximal clique C containing u there is a unique vertex of C adjacent to v. Thus the local graph Δ is the point graph of a $\mathrm{GQ}(*, \mu'-1)$ with all vertices regular.

(ii) follows directly from the two way counting of edges out of C and (iii) is straightforward. ∎

The following result is a direct consequence of Lemma 4.2 and [16, Theorem 4.3].

Theorem 4.3 *Let Γ be a distance-regular graph.*

(i) If Γ is an antipodal tight graph $\mathrm{AT4}(q, q, q)$ with $q \geq 2$, then a local graph Δ of Γ is the point graph of a generalized quadrangle $\mathrm{GQ}(q+1, q-1)$, with all its points regular.

(ii) If the graph Γ has the same intersection array as the Patterson graph, then a local graph Δ of Γ is the point graph of the generalized quadrangle $\mathrm{GQ}(3, 9)$, with all its points regular. ∎

Corollary 4.4 *Antipodal tight graphs $\mathrm{AT4}(q, q, q)$ with $q > 3$ do not exist.*

Proof. Van Maldeghem and Thas [21, Theorem 3] have shown that if a generalized quadrangle $\mathrm{GQ}(q+1, q-1)$, $q \geq 3$, has all points regular, then $q = 3$. The statement follows now directly from Theorem 4.3. ∎

Corollary 4.5 *There is no distance-regular graph with intersection array*
$\{96, 75, 24, 1; 1, 8, 75, 96\}$ *($\mathrm{AT4}(4, 4, 4)$, $v = 1288$) or*
$\{175, 144, 40, 1; 1, 10, 144, 175\}$ *($\mathrm{AT4}(5, 5, 5)$, $v = 3400$).*

Corollary 4.6 *An antipodal tight graph $\mathrm{AT4}(3, 3, 3)$, i.e., a distance-regular graph with intersection array $\{45, 32, 12, 1; 1, 6, 32, 45\}$, is unique.*

Proof. Let $\Gamma = (X, R)$ be a distance-regular graph with the above intersection array, i.e., an antipodal tight graph AT4(3, 3, 3). By Theorem 4.3, its local graphs are the point graphs of GQ(4, 2). Then the uniqueness follows by Brouwer and Blokhuis [3, p. 399]. ∎

Remark 4.7 A. E. Brouwer notes that we do not need to refer to Brouwer and Blokhuis [3, p. 399] and provides (private communication) an alternative proof. The following is based on [9, Corollary 4.5.4]:

We first show the uniqueness of the antipodal quotient by proving that it is a Zara graph with parameters (126, 6, 2), i.e., it has v vertices, every maximal clique has size 6, and for every maximal clique C and every vertex u not in C, there are exactly 2 vertices in C adjacent to u. Then it is the $O_6^-(3)$ graph by Blokhuis and Brouwer [2].

Let $\Gamma = (X, R)$ be an antipodal tight graph AT4(3, 3, 3). By Theorem 4.3, its local graphs are the point graphs of GQ(4, 2). As Γ is locally the same as its antipodal quotient, the maximal cliques in both graphs have size six. The Delsarte's clique bound (often called Hoffman's clique bound) is met in Γ, see Godsil [7, p. 276], or Brouwer et al. [3, Proposition 4.4.6], so each point not in some maximal clique has exactly zero or two neighbours in that maximal clique. We will prove that in the antipodal quotient the zero case cannot occur.

Let u and v be vertices of the quotient graph and C be a maximal clique containing u but not v. If $v \in \Gamma(u)$ then, by the property of generalized quadrangles, v has exactly one neighbour in $C \cap \Gamma(u)$, hence precisely two neighbours in C. Let us now suppose $v \in \Gamma_2(u)$. We want to prove that v has two neighbours in $C \cap \Gamma(u)$ by showing that the common neighbours of u and v consist of two disjoint ovoids of the generalized quadrangle GQ(4, 2).

A μ-graph of the antipodal quotient consists of three copies of $K_{3,3}$. By taking three independent vertices from each copy, we get the set of nine independent vertices of GQ(4, 2), which means that we have an ovoid of GQ(4, 2). The other nine vertices also correspond to an ovoid. Therefore, the vertices of μ-graph of u and v correspond to two disjoint ovoids, and thus v has precisely two neighbours in each maximal clique containing the vertex u. Thus the antipodal quotient is the desired Zara graph.

Consider its universal cover modulo triangles, cf. [4]. Since a μ-graph of this folded graph consists of three copies of $K_{3,3}$, and in particular has three connected components, the covering index of this universal cover is at most 3 (because triangulated connected components stay connected). Since such a graph actually exists, the covering index of this universal cover is 3. Our graph Γ is a quotient of this universal cover, hence it must be equal to it, and is unique, since the universal cover is unique.

5 Classification of the family AT4(qs, q, q)

In this section we will show Theorem 1.2.

Theorem 5.1 *Let Γ be an antipodal tight graph* AT4(p, q, q) *with $q \geq 3$. Then $p = qs$, where s is a positive integer, and one of the following (i)–(iii) holds.*

(i) $(s, q) = (1, 3)$ and Γ is the $3.O_6^-(3)$ graph (AT4$(3, 3, 3)$ *).*

(ii) $(s, q) = (2, 4)$ and Γ is locally strongly regular with parameters $(176, 40, 12, 8)$, locally² GQ$(3, 3)$, and

(iii) $(s, q) = (3, 3)$ and Γ is locally strongly regular with parameters $(117, 36, 15, 9)$, locally² strongly regular with parameters $(36, 15, 6, 6)$, locally³ GQ$(2, 2)$.

Remark 5.2 The Meixner2 graph is one of the examples of AT4$(8, 4, 4)$ in (ii) and the $3.O_7(3)$ graph is one of the examples of AT4$(9, 3, 3)$ in (iii). As we have mentioned in Introduction, we used the above information to prove their uniqueness.

Proof. As we mentioned in (1), cf. [11, Corollary 3.5], we have $r \,|\, p + q$ and hence $q \,|\, p$, i.e., $p = qs$ for $s \in \mathbb{N}$.

Case $s = 1$. (i) follows immediately by Corollaries 4.6 and 4.4.

So we assume $s \geq 2$. Hence, the μ-graphs of Γ are equal to $K_{t \times n}$, where $t = s + 1$ and $n = q$, by Theorem 1.1, The local graph $\Delta(x)$ of Γ is strongly regular, $v'' = k' = sq(q + 1)$, $k'' = \lambda' = (2s - 1)q$, $\mu' = qs$, $\mu'' = q(s - 1)$ and $\alpha = s + 1$ by Proposition 2.1(ii). Let x and y be two adjacent vertices of Γ and $z \in \Gamma(x, y)$.

Case $s = 2$. Then $\alpha = 3$ and the μ-graphs of Γ are $K_{n,n,n}$. The local graph $\Delta(x)$ has $\alpha = 2$ and its μ-graphs are the complete bipartite graph $K_{n,n}$, so we can apply Lemma 4.2 to $\Delta(x)$ and its local graph $\Delta(x, y)$, i.e., the local graph of y in $\Delta(x)$. Let c be the size of a maximal clique of $\Delta(x, y)$. Then we have

$$2q(q + 1) = v'' = c(k'' - c + 2) = c(3q + 2 - c)$$

and thus $c \in \{q, 2(q + 1)\}$. The vertex z is contained in $\mu'' = q$ maximal cliques of $\Delta(x, y)$ and its valency is $k'' = 3q$. This is possible only for $c = q = 4$, which means that all maximal cliques have size four, $\Delta(x, y)$ is the point graph of a generalized quadrangle GQ$(3, 3)$ and (ii) follows.

Let us now assume $s \geq 3$. Then $t = s + 1 \geq 4$. Obviously we have also $a_2 = pq^2 = sq^3 \neq 0$ and $n = q \geq 3$, so we can apply Theorem 3.1(iii). The following expression

$$\kappa := \alpha - 2 + (n - 1)((t - 3)n - \alpha + 3) = (s - 1) + (q - 1)^2(s - 2)$$

is a lower bound on the valency of the subgraph $\Delta(x, y, z)$. By inequality of Theorem 3.1(ii), we obtain

$$(s - 1)\left[q\big(s(q + 1) - (2s - 1)\big) - 1\right] \leq (2s - 1)\left[(2s - 1)q - (q - 1)^2(s - 2) - s\right] \qquad (4)$$

In the above inequality the LHS is an increasing and the RHS is a nonincreasing function in terms of $q \geq 3$ and for $s \geq 3$, hence we set $q = 3$ and obtain $-10s^2 + 37s - 21 \geq 0$. It follows $s = 3$.

Case $s = 3$. The inequality (4) simplifies to $q(27 - 7q) \geq 18$ and so $q = 3$. We have equality in (4), which means that the subgraph $\Delta(x, y)$ is a strongly regular graph by Theorem 3.1(ii). Its parameters are then $(v'', k'', \lambda'', \mu'') = (36, 15, 6, 6)$.

We are going to show that the subgraph $\Delta(x, y, z)$ is the point graph of a generalized quadrangle GQ(2, 2). In order to use Lemma 4.2(ii) applied to the subgraph $\Delta(x, y)$, we first note that $\alpha(\Delta(x, y)) = 2$ and the μ-graphs of $\Delta(x, y)$ are $K_{3,3}$. By Lemma 4.2(ii) applied to the subgraph $\Delta(x, y)$, a maximal clique C in $\Delta(x, y, z)$ with size c satisfies $c(8 - c) = 15$, therefore $c = 3$ or $c = 5$. On the other hand each vertex of $\Delta(x, y, z)$ has valency $k''' = \lambda'' = 6$ and is in $\mu''' = 3$ maximal cliques, so $c = 5$ is impossible. We have shown that each maximal clique of $\Delta(x, y, z)$ has size 3, therefore, by Theorem 3.1 applied to $\Delta(x, y)$, its local graph is the point graph of a generalized quadrangle GQ(2, 2). Finally, also (iii) and the whole statement follow. ∎

Corollary 5.3 *There is no distance-regular graph with intersection array*

- $\{81, 56, 18, 1; 1, 9, 56, 81\}$ (AT4(6, 3, 3), $v = 750$),
- $\{189, 128, 36, 1; 1, 18, 128, 189\}$ (AT4(15, 3, 3), $v = 1914$),
- $\{336, 255, 60, 1; 1, 20, 255, 336\}$ (AT4(16, 4, 4), $v = 5632$) *or*
- $\{416, 315, 72, 1; 1, 24, 315, 416\}$ (AT4(20, 4, 4), $v = 7128$).

Remark 5.4 Note that AT4(6, 3, 3) was the smallest open case that the conjecture mentioned in Introduction claimed that it does not exist. The next two smallest open case are AT4(8, 4, 3) and AT4(9, 6, 3).

Proof of Theorem 1.2. If Γ belongs to one of the items (i)–(vi), then for each graph there exists $n, t \in \mathbb{N}\backslash\{1\}$ such that its μ-graphs are complete multipartite graphs $K_{t\times n}$ by [16, Theorem 4.3]. Let us now prove the converse. By [14, Theorem 6.3] (cf. [15, Theorem 5.3]), we have one of the cases (i), (ii) or (iii) for $q \leq 2$, so we may assume $q \geq 3$. By Theorem 1.1, there exists an integer s such that $p = qs$, $r = q$. Therefore, $n = q$, $t = s + 1$ and $\alpha = s + 1$ by Propositions 2.1. Hence $\alpha = t$ and we have one of the cases (iv), (v) or (vi) by Theorems 1.1 and 5.1. ∎

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