tion numbers of association schemes discovered by Scott [117] and the *absolute bound* discovered by Neumaier [106].

2.3.2 THEOREM (Krein condition). *Let G be a distance-regular graph with n vertices, diameter d and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$ with multiplicities $m_0, \ldots, m_d$. Let the polynomials $v_i(x)$ and the numbers k_i be as above. Then the Krein parameters (also called the dual intersection numbers)*

$$q_{ij}(h) = \frac{m_i m_j}{n} \sum_{a=0}^{d} \frac{v_a(\theta_i) v_a(\theta_j) v_a(\theta_h)}{k_a^2}$$

are nonnegative for all $i, j, h \in \{0, \ldots, d\}$. □

2.3.3 THEOREM (Absolute bound). *Let G be a distance-regular graph of diameter d. Then the multiplicities $m_0, \ldots, m_d$ of its eigenvalues satisfy*

$$\sum_{q_{ij}(h) \neq 0} m_h \leq \begin{cases} \frac{1}{2} m_i (m_i + 1) & \text{if } i = j \\ m_i m_j & \text{if } i \neq j \end{cases}$$

where the $q_{ij}(h)$ are the Krein parameters. □

4. Antipodal distance-regular graphs

Let G be a graph with a partition π of its vertices into cells satisfying the following conditions:

(a) each cell is an independent set,

(b) between any two cells there are either no edges or there is a matching.

Let G/π be the graph with the cells of π as vertices and with two of them adjacent if and only if there is a matching between them. Then we say that G is a *cover* of G/π and we call the cells and the matchings the *fibres of vertices* and the *fibres of edges* respectively. If G/π is connected, then all cells have the same size which is called the *index* of the cover, and is denoted by r. In this case G is called an r-cover of G/π. In this thesis we will always require that $r > 1$.

We can give an equivalent definition of a cover H of G using the projection map p from $V(H)$ to $V(G)$. We say that H is a cover of G if there is a map $p : V(H) \to V(G)$ called a *projection* which is a graph morphism, i.e., preserves adjacency, and a local isomorphism, i.e., for each vertex u of H the map p restricted to $\{u\} \cup S(u)$ is bijective. Then $\{p^{-1}(u), u \in G\}$ is the set of fibres and $r = |p^{-1}(u)|$ is the index of the covering. If we consider our graphs as simplicial complexes, coverings graphs are covering spaces in the usual topological sense.

If a graph G is a cover of G/π and π consists of its antipodal classes, then G is called an *antipodal cover*. Furthermore, if the graph G is also distance-regular, we say that G is a *distance-regular antipodal cover*.

2.4.1 LEMMA. *A distance-regular antipodal graph G of diameter d is a cover of its antipodal quotient with components of G_d as its fibres unless $d = 2$.*

To prove the above result we need only the facts that G is antipodal, connected and that $b_{d-1}(u, v) > 0$ for any vertex u and $v \in S_{d-1}(u)$.

In order to gain more insight into the structure of the distance-regular antipodal covers of distance-regular graphs let us first prove the following extension of a result due to Gardiner [60]. The part (i) is new, and (ii) modified, however the proofs of (i)⇒(ii) and (i)&(ii)⇒(iii) are motivated by his proof.

For each vertex u of a cover H we denote the fibre which contains u by $F(u)$. A *geodesic* in a graph G is a path $g_0, \ldots, g_t$, where $\mathrm{dist}(g_0, g_t) = t$.

2.4.2 THEOREM. *Let G be a distance-regular graph of diameter d with parameters b_i, c_i and H its r-cover of diameter $D > 2$. Then the following statements are equivalent:*

(i) The graph H is antipodal with its fibres as the antipodal classes (hence an antipodal cover of G) and each geodesic of length at least $\lfloor (D+1)/2 \rfloor$ in H can be extended to a geodesic of length D.

(ii) For any $u \in V(H)$ and $i \in \{0, 1, \ldots, \lfloor D/2 \rfloor\}$ we have

$$S_{D-i}(u) = \cup\{F(v)\backslash\{v\} \ : \ v \in S_i(u)\}.$$

(iii) The graph H is distance-regular with $D \in \{2d, 2d+1\}$ and intersection array

$$\left\{b_0, \ldots, b_{d-1}, \tfrac{(r-1)c_d}{r}, c_{d-1}, \ldots, c_1; c_1, \ldots, c_{d-1}, \tfrac{c_d}{r}, b_{d-1}, \ldots, b_0\right\}$$

for D even, and

$$\left\{b_0, \ldots, b_{d-1}, (r-1)t, c_d, \ldots, c_1; c_1, \ldots, c_d, t, b_{d-1}, \ldots, b_0\right\}$$

for D odd and some integer t.

Proof. Let H be an antipodal cover. If two paths both have length less than D and they go through the same fibres in the same order, then we will say that they are *parallel*. Note that two parallel paths have the same length and that one of them and a vertex from the other one uniquely determine the other path. By antipodality, a path of length less than D contains at most one vertex from each fibre, therefore two parallel paths are either disjoint or equal corresponding to their intersection being empty or not. Finally, the parallelism is an equivalence relation, each parallel class corresponds bijectively to a path in the antipodal quotient of H, and each parallel class contains r elements. The last, for example, implies that for two distinct fibres any vertex from them lies in a shortest path between them (cf. [27, Lemma 11.1.4]).

(i)⇒(ii): Let u and v be any two vertices of H which are at distance $i \leq \lfloor D/2 \rfloor$. Since $F(v)$ is an antipodal class, the distance from u to any vertex of $F(v)$ is at least i. Let P be a path of length i between u and v. Then P is a shortest path between $F(u)$ and $F(v)$. Note that the set of all ends of paths from the parallel class of P equals $F(u) \cup F(v)$, and consider the distance partition corresponding to u. Let P' be a parallel path of P which has one end in $S_D(u)$ and the other end in $S_j(u)$ for some $j \geq D - i$. The required property of geodesics implies the existence of a path of length $D - j$ between $V(P') \cap S_j(u)$ and $S_D(u)$. But this is also a path between $F(v)$ and $F(u)$, so $D - j \geq i$. Therefore $j = D - i$ and $S_D(v) \subseteq S_{D-i}(u)$. Now let w be any vertex in $S_{D-i}(u)$. Then the extension of a geodesic from u to w to $S_D(u)$ is a shortest path between $F(u)$ and $F(w)$ and a path from its parallel class starting at u has to end in $S_i(u)$. Hence $\cup\{S_D(v) : v \in S_i(u)\} \subseteq S_{D-i}(u)$.

(i)⇐(ii): $i = 0$ implies that the graph H is antipodal with its fibres as the antipodal classes and therefore $S_{D-i}(u) = \cup\{S_D(v) \ : \ v \in S_i(u)\}$. The rest is now straightforward.

(i)&(ii)⇒(iii): A geodesic P of length d corresponds to a parallel class of geodesics of length $d < D$. These are the shortest paths between two fibres since P is a geodesic. Therefore by (ii) $D \geq 2d$. If $D \geq 2d + 2$ then by (ii) there exists a geodesic in H of length $d + 1$ which is the shortest path between two fibres and therefore $\mathrm{diam}(G) \geq d + 1$. Contradiction! The remainder of this part of the proof is only sketched. Suppose $D = 2d$ and let $\{u_1, \ldots, u_r\}$ be an antipodal class of H. Then the balls $B_{d-1}(u_i)$ of radius $d - 1$ centered at u_i (i.e., $\{u_i\} \cup S_1(u_i) \cup \ldots \cup S_{d-1}(u_i)$) for $i = 1, \ldots, r$ are disjoint and there are no edges between any two of them. Their induced graphs are parallel in the above sense and therefore isomorphic to their projection. This implies the desired parameters of H. The case when D is odd can be treated similarly.

(i)⇐(iii) It suffices to prove that, for a vertex $u \in V(H)$, any two distinct vertices v and w in $S_D(u)$ are at distance at least D. Suppose that $B_i(v) \cap B_i(w) = \emptyset$ and that there are no edges between $B_{i-1}(v)$ and $B_{i-1}(w)$ for some

$i \in \{1, \ldots, \lfloor D/2 \rfloor\}$. This is certainly satisfied for $i = 1$, since $a_D(H) = 0$ and $b_{D-1}(H) = 1$. If $i = (D-1)/2$ our work is done, otherwise $a_{D-i}(H) = a_i(H)$ implies that there is no edges between $S_i(v)$ and $S_i(w)$. If $i = D/2$ our work is done again, otherwise by $c_{D-i-1}(H) = b_{i+1}(H)$ the sets $S_{i+1}(v)$ and $S_{i+1}(w)$ are disjoint. So the induction assumption is satisfied for $i + 1$. □

Remarks: Statement (ii) gives us an idea how to draw the distance partition of an antipodal cover over the corresponding distance partition of its antipodal quotient and why we say that a distance-regular antipodal cover folds to its antipodal quotient (see Figures 3.1, 4.1 and 4.3).

(i) In $D = 2d$ case the integrality of entries in the intersection array implies $r|c_d$. By the monotonicity of parameters B_i and C_i there is also $c_{d-1} \leq \frac{c_d}{r}$ and $(1 - \frac{1}{r})c_d \leq b_{d-1}$.

(ii) In $D = 2d + 1$ case the integer t satisfies the conditions $t(r - 1) \leq \min(b_{d-1}, a_d)$ and $c_d \leq t$.

The following corollary can again be found in Gardiner [60].

2.4.3 COROLLARY. *If H is a distance-regular antipodal graph, then H has a distance-regular antipodal cover only if H is either a cycle, a complete graph or a complete bipartite graph.* □

In the reminder of this section we determine also the eigenvalues of antipodal distance-regular graphs and their multiplicities.

2.4.4 THEOREM. *Let G be a distance-regular graph and H a distance-regular antipodal r-cover of G. Then every eigenvalue θ of G is also an eigenvalue of H with the same multiplicity.* □

The above result can be proved by combining the properties of the antipodal partition of H and the quotient graph of H, but it can also be derived as a consequence of the following theorem of Biggs [14]:

2.4.5 THEOREM. *The multiplicity of an eigenvalue θ of a distance-regular graph G with diameter d and n vertices is equal to*

$$\frac{n}{\sum_{i=0}^{d} k_i w_i(\theta)^2} .$$

□

Now we can finally state a result due to Biggs and Gardiner [16]:

2.4.6 THEOREM. *Let H be a distance-regular antipodal r-cover with diameter D of the distance-regular graph G with diameter d and parameters a_i, b_i, c_i. The $D-d$ eigenvalues of H which are not eigenvalues of G are, in the case when $D = 2d$, the eigenvalues of the $d \times d$ matrix*

$$\begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-2} & a_{d-2} & b_{d-2} \\ & & & & c_{d-1} & a_{d-1} \end{pmatrix}$$

and, in the case when $D = 2d+1$, the eigenvalues of the $(d+1) \times (d+1)$ matrix

$$\begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d - rt \end{pmatrix}$$

If $\theta_0 \geq \theta_1 \geq \cdots \geq \theta_D$ are the eigenvalues of H and $\xi_0 \geq \xi_1 \geq \cdots \geq \xi_d$ are the eigenvalues of G, then

$$\xi_0 = \theta_0, \;\; \xi_1 = \theta_2, \;\; \cdots, \;\; \xi_d = \theta_{2d}$$

i.e., the eigenvalues of G interlace the 'new' eigenvalues of H. □

Thus, in the even case the new eigenvalues do not depend on r and are the roots of $w_d(\theta) = 0$. Their multiplicities are proportional to $r-1$. In the odd diameter case the new eigenvalues depend only on rt and are the roots of $c_d w_{d-1}(\theta) + w_d(\theta)(a_d - rt - \theta) = 0$.