A Local Approach to 1-Homogeneous Graphs

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Abstract

Let Γ be a distance-regular graph with diameter d. For vertices x and y of Γ at distance i, $1 \le i \le d$, we define the sets $C_i(x,y) = \Gamma_{i-1}(x) \cap \Gamma(y)$, $A_i(x,y) = \Gamma_i(x) \cap \Gamma(y)$ and $B_i(x,y) = \Gamma_{i+1}(x) \cap \Gamma(y)$. Then we say Γ has the CAB_j property, if the partition $CAB_i(x,y) = \{C_i(x,y), A_i(x,y), B_i(x,y)\}$ of the local graph of y is equitable for each pair of vertices x and y of Γ at distance $i \le j$. We show that if Γ has the CAB_j property then the parameters of the equitable partitions $CAB_i(x,y)$ do not depend on the choice of vertices x and y at distance i for all $i \le j$. The graph Γ has the CAB property if it has the CAB_d property. We show the equivalence of the CAB property and the 1-homogeneous property in a distance-regular graph with $a_1 \ne 0$. Finally, we classify the 1-homogeneous Terwilliger graphs with $c_2 \ge 2$.

1 Introduction

In this paper we take a local approach to the 1-homogeneous property. We will do this via the CAB_j property defined below. Before we summarize our main results we establish some notation and review basic definitions, for more details see Brouwer, Cohen and Neumaier [2], and Godsil [5].

Let us first recall that an **equitable partition** of a graph is a partition $\pi = \{C_1, \ldots, C_s\}$ of its vertices into cells, such that for all i and j the number c_{ij} of neighbours, which a vertex in C_i has in the cell C_j, is independent of the choice of the vertex in C_i. In other words, the partition π is equitable when each cell C_i induces a regular graph of valency c_{ii}, and between any two cells C_i and C_j there is a biregular graph, with vertices of the cells C_i and C_j having valencies c_{ij} and c_{ji} respectively.

Let Γ be a connected graph with diameter d. For a vertex u of Γ we define $\Gamma_i(u)$ to be the set of vertices at distance i from u, and denote $\Gamma_1(u)$ by $\Gamma(u)$ and the cardinality of $\Gamma_i(u)$ by k_i. Then one way to define **distance-regularity** of Γ is to say that for all $i \in \{0, \ldots, d\}$, and vertices x and y of Γ at distance i, the cardinalities c_i, a_i and b_i of the sets

$$C_i(x,y) = \Gamma_{i-1}(x) \cap \Gamma(y), \quad A_i(x,y) = \Gamma_i(x) \cap \Gamma(y) \quad \text{and} \quad B_i(x,y) = \Gamma_{i+1}(x) \cap \Gamma(y),$$

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respectively, are independent of the choice of vertices x and y at distance i. Alternatively, Γ is a distance-regular graph when the distance partition corresponding to any vertex x of Γ is equitable and the parameters of the equitable partition do not depend on x. Godsil and Shawe-Taylor [6] defined a graph Γ to be distance-regularised if we require only that the distance partition corresponding to any vertex x of Γ is equitable, and showed that distance-regularised graphs are either biregular or regular, in which case they are distance-regular.

A generalization of the distance-regularity property is the i-homogeneous property. For all vertices x and y of Γ at distance i and integers j and h we define $D^h_j(x,y) = \Gamma_j(x) \cap \Gamma_h(y)$. Nomura defined in [13] that Γ has ***i*-homogeneous property** when the distance partition corresponding to any pair x, y of vertices at distance i, i.e., the collection of nonempty sets $D^j_h(x,y)$, see Figure 3.1, is equitable, and the parameters corresponding to equitable partitions are independent of vertices x and y at distance i. The definition of the 0-homogenous property coincides with the definition of distance-regularity.

Let Γ be a distance-regular graph with diameter d. For vertices x and y of Γ at distance i let $\mathbf{CAB}_i(x,y)$ be the partition $\{C_i(x,y), A_i(x,y), B_i(x,y)\}$ of the neighbourhs of x. Let us define the **local graph** $\Delta(x)$ as the subgraph of Γ, induced by the neighbours of x. We say that Γ has the $\mathbf{CAB}_j$ **property**, if for each $i \le j$ the partition $\mathrm{CAB}_i(x,y)$ of $\Delta(y)$ is equitable for each pair of vertices x and y of Γ at distance i. Then the graph Γ has the CAB_1 property if and only if its local graphs are strongly regular.

The main result of Section 2 is that in the graph Γ with the CAB_j property the parameters of the equitable partitions $\mathrm{CAB}_i(x,y)$, $i \le j$, do not depend on the vertices x and y at distance i. We say the graph Γ has the **CAB property** if it has the CAB_d property. Locally disconnected graphs having the CAB property turn out to be precisely regular near $2d$-gons, while locally disconnected graphs having the CAB_{d-1} property turn out to be precisely regular near polygons, cf. Nomura [13, Thm. 1]. The main result of Section 3 is the equivalence of the CAB property and 1-homogeneous property for a distance-regular graph with $a_1 \ne 0$. For a distance-regular graph with diameter $d \ge 2$, $a_1 \ne 0$ and the CAB property we determine recursively all the intersection numbers of Γ and all the parameters that correspond to equitable partitons CAB_i, $i \le d$ in terms of the eigenvalues of a local graph and intersection numbers $c_2, \ldots, c_{d-1}$.

In Section 4 we study 1-homogeneous Terwilliger graphs. A distance-regular graph with diameter $d \ge 2$ is a **Terwilliger graph** if and only if it contains no induced quadrangles. Let Γ be a 1-homogeneous Terwilliger graph with diameter $d \ge 2$. Then $c_2 = 1$ and the local graphs are disconnected, or $c_2 = 2$ and we show that all the local graphs of Γ are Moore graphs with diameter two. We give a finite algorithm for determining all possible intersection arrays of 1-homogeneous graphs, whose local graphs are strongly regular with given parameters. We apply this algorithm to classify 1-homogeneous graphs whose local graphs are Moore graphs with diameter two. Therefore, we obtain a classification of the 1-homogeneous Terwilliger graphs with $c_2 \ge 2$.

2 The CAB property

In this section we develop some basic properties of a distance-regular graph Γ having the CAB_j property. We start by showing that for $j = 1$ the parameters of the equitable partition $\mathrm{CAB}_1(x, y)$ do not depend a pair of adjacent vertices x and y, and thus all local graphs are strongly regular with the same parameters. Then we turn our attention to the case when Γ is locally disconnected (for example when $c_2 = 1$) with $d \geq 2$, $a_1 \neq 0$, $j \in \{d-1, d\}$, and show that in this case Γ is a regular near polygon for $j = d-1$, and a regular near $2d$-gon for $j = d$. The main result of this section is that in Γ with diameter d and the CAB_j property for some $j \in \{1, \ldots, d\}$ the paramters of the equitable partition $\mathrm{CAB}_i(x, y)$ do not depend on the choice of x and y, but only on their distance. We determine all the parameters corresponding to the equitable partitions CAB_i of Γ, $i \leq j$, in terms of the intersection numbers $b_0, \ldots, b_i$ and $c_2, \ldots, c_i$. We conclude this section by determining recursively all of the parameters that correspond to the CAB_i partition for $i \leq j$ and $j \neq d$ in terms of the eigenvalues of a local graph and intersection numbrs $c_2, \ldots, c_j$. (It remains to show this also for $j = d$, but this will become trivial once we will determine c_d and γ_d.)

In order to be more precise we repeat the definitions of the CAB properties. Let Γ be a distance-regular graph with diameter d, $a_1 \neq 0$ and let $i \in \{1, \ldots, d-1\}$. Then $a_i \neq 0$ by [2, Prop. 5.5.1]. Let us call the partition $\{C_i(x, y), A_i(x, y), B_i(x, y)\}$ of the local graph $\Delta(y)$ the **$\mathrm{CAB}_i(x, y)$ partition**. The **$\mathrm{CAB}_d(x, y)$ partition** is the partition $\{C_d(x, y), A_d(x, y)\}$ when $a_d \neq 0$ and $\{C_d(x, y)\}$ when $a_d = 0$. We say that Γ has the **CAB_j property** for some $j \in \{1, \ldots, d\}$, if for each $i \leq j$ the $\mathrm{CAB}_i(x, y)$ partition of $\Delta(y)$ is equitable for each pair of vertices x and y of Γ at distance i. Let us say Γ has the **strong CAB_j property** if it has the CAB_j property and for each $i \leq j$ the parameters of the equitable partition $\mathrm{CAB}_i(x, y)$ do not depend on choice of vertices x and y of Γ at distance i. As we have already mentioned, the main result of this section is that the CAB_j property implies the strong CAB_j property. The graph Γ has the **(strong) CAB property** if it has the (strong) CAB_d property.

All CAB_i partitions are partitions of local graphs. We know that every local graph has $k = b_0$ vertices and valency a_1. The $\mathrm{CAB}_1(x, y)$ partition is equal to $\{D_1^0(x, y), D_1^1(x, y), D_1^2(x, y)\}$, see Figure 3.1. If Γ is locally connected, then the partition $\mathrm{CAB}_1(x, y)$ is the distance partition of the local graph $\Delta(y)$ corresponding to the vertex x. Therefore, if Γ has 1-homogeneous property, then its local graphs are strongly regular. In the case a local graph Δ is strongly regular we will denote its parameters $v(\Delta)$, $k(\Delta)$, $a_1(\Delta)$, $c_2(\Delta)$ resp. by v', k', λ', μ' resp., and when Δ is also connected, we will denote its nontrivial eigenvalues by p and q, and assume $p > q$.

Proposition 2.1 *Let Γ be a distance-regular graph with $a_1 \neq 0$ and the CAB_1 property. Then all the local graphs of Γ are either*

(i) connected strongly regular graphs with the same parameters, or

(ii) disjoint unions of (a_1+1)-cliques,

and Γ has the strong CAB_1 property.

Proof. The local graphs have size $v' = k$ and valency $k' = a_1$. If $v' = k' + 1$, then the local graphs are complete graphs, Γ is the complete graph K_{k+1} and the statement is evidently true. Let us now suppose $v' > k'+1$. Let us consider a pair of adjacent vertices x, y in Γ and the $\mathrm{CAB}_1(x, y)$ partition. Then $C_1(x, y) = \{x\}$, $A_1(x, y) \neq \emptyset$, and each element of $A_1(x, y)$ is adjacent to the vertex x. Let us denote the valency of the subgraph of Γ, induced by the set $A_1(x, y)$, by $\lambda'(x, y)$. Then any two edges of the local graph $\Delta(y)$ that share a vertex, say w, lie in the same number of triangles of $\Delta(y)$ (this number is $\lambda'(w, y)$). Hence, each edge, which is in the same connected component of $\Delta(y)$ as x, lies in exactly $\lambda'(x, y)$ triangles of $\Delta(y)$.

Suppose $\lambda'(x, y) < k' - 1$. Then the local graph $\Delta(y)$ is connected. Let us denote $\lambda'(x, y)$ by λ', since it does not depend on x and y. Then the number $\mu'(x, y)$ of neighbours each vertex of $B_1(x, y)$ has in $A_1(x, y)$ equals

$$\mu' = \frac{k'(k' - 1 - \lambda')}{v' - k' - 1} = \frac{a_1(a_1 - 1 - \lambda')}{k - a_1 - 1}.$$

Since the same is true if we replace x by any vertex of $\Delta(y)$, the local graph $\Delta(y)$ is a connected strongly regular graph with parameters (v', k', λ', μ'). But then also the local graph $\Delta(x)$ is connected strongly regular graph with parameters (v', k', λ', μ'). Finally, by the connectivity of Γ, all its local graphs are connected strongly regular graphs with parameters (v', k', λ', μ').

It remains to consider the case when $\lambda'(x, y) = k' - 1$ for all adjacent vertices x and y of Γ. This implies the set $A_1(x, y) \cup \{x\}$ induces a clique, which is a connected component in $\Delta(y)$, for each x in $\Delta(y)$ and for each y in Γ. Therefore, all the local graphs are disjoint unions of cliques on $k' + 1 = a_1+1$ vertices. ∎

Let us now consider distance-regular graphs with the CAB_j property, $j \geq 2$, in the case when they are locally disconnected.

Proposition 2.2 *Let Γ be a distance-regular graph with diameter $d \geq 2$, $a_1 \neq 0$ and the CAB_j property for some $j \geq 2$. Then the following are equivalent.*

(i) *The graph Γ is locally disconnected.*

(ii) *The set $C_i(x, y)$ is independent in Γ for any vertices x and y of Γ at distance i and for all $i \leq j$.*

(iii) $a_i = c_i a_1$ *for all* $i \leq j$.

Proof. (i) $\Longrightarrow$ (ii),(iii): Since the graph Γ is locally disconnected, the local graphs of Γ are disjoint unions of cliques on $a_1 + 1$ vertices and $\mu' = 0$ by Proposition 2.1. We will prove, by induction on i, that for any vertices x and y of Γ at distance i, $i \leq j$, the set $C_i(x, y)$ is independent in Γ and that this implies $a_i = c_i a_1$. The statement is obviously true for $i = 1$, as $C_1(x, y) = \{x\}$ and $c_1 = 1$.

Let us now assume for some $i \leq j$ that for all vertices u and v in Γ at distance i the set $C_i(u, v)$ is independent in Γ. This implies that each vertex in $C_i(u, v)$ has a_1 neighbours in $A_i(u, v)$. Furthermore, each vertex in $A_i(u, v)$ has exactly one neighbour in $C_i(u, v)$, since $a_1 \neq 0$ and $\Delta(v)$ is a disjoint union of cliques. Therefore, $c_i a_1 = a_i$ and there are no edges between the sets $A_i(u, v)$ and $B_i(u, v)$.

It remains to prove that for all vertices x and y of Γ at distance $i+1$ the set $C_{i+1}(u,v)$ is independent if $i<j$. Let $z \in C_{i+1}(x,y)$. Then $y \in B_i(x,z)$ and since there are no edges between the sets $A_i(x,z)$ and $B_i(x,z)$, the set

$$\Gamma(x) \cap A_i(x,z) = \Gamma(y) \cap \Gamma(z) \cap \Gamma_i(x) = \Gamma(z) \cap C_{i+1}(x,y), \qquad (1)$$

is empty, which means the set $C_{i+1}(x,y)$ is independent.

(ii)$\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i): Let us prove that the negation of (i) implies the negations of (ii) and (iii). Let us assume that Γ is locally connected, i.e., $\mu' \neq 0$ by Proposition 2.1. Then, by [11, Theorem 3.1], the set $C_2(x,y)$ induces a graph of valency μ' and is thus not independent. For vertices x and y at distance two, each vertex in $C_2(x,y)$ has $a_1-\mu'$ neighbours in $A_2(x,y)$ by [11, Theorem 3.1]. If $a_1=\mu'$ then $a_2=0$ and this is not equal to c_2a_1. If $a_1 \neq \mu'$, then each vertex in $A_2(x,y)$ has at least one neighbour in $C_2(x,y)$, therefore, $c_2a_1 > c_2(a_1-\mu') \geq a_2$ and so $a_2 \neq c_2a_1$. ∎

Theorem 2.3 *Let Γ be a locally disconnected distance-regular graph with diameter $d \geq 2$ and $a_1 \neq 0$. Then Γ is*

(i) a regular near polygon when it has the CAB_{d-1} property, and

(ii) a regular near $2d$-gon when it has the CAB_d property.

Proof. Let $j \in \{d-1,d\}$ and let Γ have the CAB$_j$ property. Since the graph Γ is locally disconnected, it contains no $K_{1,2,1}$ by Proposition 2.1. If $j=1$, then $j=d-1$, $a_1=c_1a_1$ and Γ is a regular near polygon. Suppose $j \geq 2$. Then, by Proposition 2.2, $a_i=c_ia_1$ for $i=1,\dots,j$. Therefore, by [2, Thm. 6.4.1], Γ is a regular near polygon when $j=d-1$, and a regular near $2d$-gon when $j=d$. ∎

Theorem 2.4 *Let Γ be a distance-regular graph with diameter d, $a_1 \neq 0$ and the CAB_j property for some $j \in \{1,\dots,d\}$. Then Γ has the strong CAB_j property and the corresponding quotient matrices (see Figure 2.1) are*

$$Q_i = \begin{pmatrix} \gamma_i & a_1-\gamma_i & 0 \\ \alpha_i & a_1-\beta_i-\alpha_i & \beta_i \\ 0 & \delta_i & a_1-\delta_i \end{pmatrix}, \qquad \text{for } 1 \leq i \leq j,\ i \neq d,$$

and when $j=d$ also $Q_d = \begin{pmatrix} \gamma_d & a_1-\gamma_d \\ \alpha_d & a_1-\alpha_d \end{pmatrix}$, if $a_d \neq 0$, and $Q_d=(\gamma_d)$, if $a_d=0$, where

(i) for a locally disconnected Γ we have $\alpha_i=1$, $\beta_i=\gamma_i=\delta_i=0$ for all $i \leq j$, $i \neq d$, and when $j=d$ also $\gamma_d=0$ and $\alpha_d=1$.

(ii) for a locally connected Γ and $\delta_0=0$ we have $a_i \neq 0$, $a_1-\gamma_i \neq 0$,

$$\gamma_i=\delta_{i-1} \qquad \alpha_i=\frac{(a_1-\delta_{i-1})c_i}{a_i}, \qquad \delta_i=\frac{a_i\mu'}{a_1-\delta_{i-1}}, \qquad \beta_i=b_i\delta_i/a_i, \qquad \text{for } i \in \{1,2,\dots,j\}\backslash\{d\},$$

and when $j=d$ also $\gamma_d=\delta_{d-1}$, $\alpha_d=(a_1-\delta_{d-1})c_d/a_d$, if $a_d \neq 0$, and $\gamma_d=a_1$, if $a_d=0$.

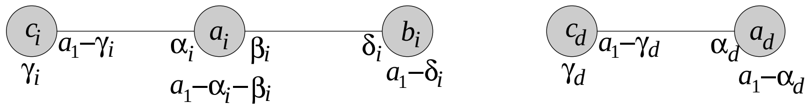

Figure 2.1: The CAB_i partition for $1 \le i \le j$, $j \ne d$, and the CAB_d partition when $a_d \ne 0$.

Proof. If the graph Γ is locally disconnected, then the statement follows directly from Propositions 2.1 and 2.2. Let us now suppose Γ is locally connected. Let $i \le j$ and let vertices x, y of Γ be at distance i. If $i = 1$, then $C_1(x,y) = \{x\}$, hence $\gamma_1 = 0 = \delta_0$. Let $i \ne d$, then for $z \in B_i(x,y)$ we have $y \in C_{i+1}(x,z)$ and all the neighbours of y in $\Gamma_i(x)$ are in $A_i(x,y)$. The set of neighbours of z in $A_i(x,y)$ is equal to the set of neighbours of y in $C_{i+1}(x,z)$, cf. (1), hence $\delta_i = \gamma_{i+1}$. The distance from a vertex in $B_i(x,y)$ to any vertex in $C_i(x,y)$ is two. By counting all the paths of length two starting in a fixed vertex of $B_i(x,y)$ and ending in $C_i(x,y)$, and noting that the ends of these paths in $C_i(x,y)$ cover all the vertices of $C_i(x,y)$ exactly μ' times, we obtain

$$\delta_i \alpha_i = c_i \mu', \tag{2}$$

By a two way counting of edges between the sets $B_i(x,y)$ and $A_i(x,y)$ and between the sets $A_i(x,y)$ and $C_i(x,y)$, we obtain

$$\delta_i b_i = a_i \beta_i, \qquad (a_1 - \gamma_i) c_i = \alpha_i a_i. \tag{3}$$

If $i = d$, then by two way counting of edges between the sets $C_d(x,y)$ and $A_d(x,y)$, we obtain and $c_d(a_1 - \gamma_d) = \alpha_d a_d$. Since Γ is locally connected, a_1, μ', $a_1 - \gamma_i = a_1 - \delta_{i-1}$, α_i, β_i, γ_{i+1} are nonzero for $1 \le i \le j$ and $i \ne d$, and if $j = d$ and $a_d \ne 0$ also $a_1 - \gamma_d = a_1 - \delta_{d-1}$ and α_d are nonzero. From (3) we get $a_i \ne 0$ and

$$\alpha_i = (a_1 - \delta_{i-1}) c_i / a_i \quad \text{and} \quad \beta_i = b_i \delta_i / a_i \qquad \text{for } \ i \in \{1, 2, \ldots, j\} \backslash \{d\}, \tag{4}$$

and if $j = d$, $a_d \ne 0$, also $\alpha_d = (a_1 - \delta_{d-1}) c_d / a_d$. This gives us, by (2), the following recursion relation on the sequence $\{\delta_i\}$:

$$\delta_i = \frac{a_i \mu'}{a_1 - \delta_{i-1}} \qquad \text{for } \ i \in \{1, 2, \ldots, j\} \backslash \{d\}. \tag{5}$$

Finally, if $j = d$ and $a_d = 0$, then $\gamma_d = a_1$. It is now clear that the sequences $\{\alpha_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$ and $\{\delta_i\}$ do not depend on the vertices x and y at distance i for all $i \le j$, but only on the entries of the intersection array of Γ. Therefore, Γ has the strong CAB_j property. ∎

Remark 2.5 *Suppose Γ in the above theorem is locally connected. By counting all the paths of length two starting in a fixed vertex of $C_i(x,y)$ and ending in $B_i(x,y)$, and noting that the ends of these paths in $B_i(x,y)$ cover all the vertices of $B_i(x,y)$ exactly μ' times, we obtain the relation $\beta_i(a_1 - \gamma_i) = b_i \mu'$. However, this relation already depends on the relations (2) and (3) (dividing this relation by (2) we get the same relation as by dividing the left relation in (3) by the right relation in (3)).*

Let Γ be a distance-regular graph with diameter d and $a_1 \neq 0$. Then the quotient graph corresponding to CAB$_i$ partition has diameter three in the case $i = \{1,\dots,j\}\backslash\{d\}$ (since $a_i \neq 0$), and diameter two in the case $i = d \neq 1$ and $a_d \neq 0$. Therefore, Q_i has in these two cases three and two distinct eigenvalues respectively. Since these eigenvalues are also the eigenvalues of the local graph, we obtain the following statement.

Lemma 2.6 *Let Γ be a locally connected distance-regular graph with diameter $d \geq 2$, $a_1 \neq 0$ and the CAB$_j$ property for some $j \in \{1,\dots,d\}$. Then the eigenvalues of the matrix Q_i are the eigenvalues a_1, p and q of a local graph, for $i = \{1,\dots,j\}\backslash\{d\}$, and when $j = d$, $a_d \neq 0$, then the eigenvalues of Q_d are a_1 and a nontrivial eigenvalue of a local graph.* ∎

Theorem 2.7 *Let Γ be a locally connected distance-regular graph with diameter $d \geq 2$, $a_1 \neq 0$ and CAB$_j$ property for some $j < d$, and let $\delta_0 = 0$. Then for $1 \leq i \leq j$ the parameters a_i, b_i, α_i, β_i, γ_i, δ_i can each be expressed in terms of the eigenvalues $a_1 > p > q$ of a local graph and the parameters δ_{i-1} and c_i.*

Proof. Since a local graph is connected and strongly regular by Proposition 2.1, we have $\mu' = k + pq$ and $k = (a_1 - p)(a_1 - q)/\mu'$, for example by Cameron, Goethals and Seidel [3]. By $a_i = k - b_i - c_i$ and Theorem 2.4 we have $\gamma_i = \delta_{i-1}$ and

$$\beta_i = \frac{\mu' b_i}{a_1 - \gamma_i}, \qquad \delta_i = \frac{a_i \beta_i}{b_i} = \frac{\mu'(k - c_i)}{a_1 - \gamma_i} - \beta_i, \qquad \alpha_i = \frac{c_i(a_1 - \gamma_i)}{a_i} = \frac{c_i(a_1 - \gamma_i)}{k - c_i - b_i}.$$

The statement is obviously true for $i = 1$, so let us assume $i > 1$. The trace $\mathrm{tr}(Q_i) = \gamma_i + a_1 - \beta_i - \alpha_i + a_1 - \delta_i$ can be expressed as a sum of the fraction $c_i(a_1 - \delta_{i-1})/(k - c_i - b_i)$ and a function $f_1(a_1, p, q, c_i, \delta_{i-1})$. On the other hand $\mathrm{tr}(Q_i) = a_1 + p + q$, by Lemma 2.6, so it follows that the parameter b_i can be expressed in terms of terms of the eigenvalues of a local graph and parameters δ_{i-1} and c_i. ∎

Remark 2.8 *(i) Under the assumptions of the above theorem we also can conclude that for $1 \leq i \leq j$ the parameters a_i, c_i, α_i, β_i, γ_i, δ_i can each be expressed in terms of the eigenvalues $a_1 > p > q$ of a local graph and the parameters δ_i and b_i.*
(ii) The fact that the determinant of Q_i equals the product of its eigenvalues gives us the following relations $(a_1 - \delta_i)(\gamma_i - \alpha_i) - \gamma_i \beta_i = \mu' - a_1$ for $i \in \{1,\dots,j\}\backslash\{d\}$, which are, by Theorem 2.4(ii), equivalent to the following relations $a_1 - \delta_i + \gamma_i - \alpha_i - \beta_i = \lambda' - \mu'$ for $i \in \{1,\dots,j\}\backslash\{d\}$ used in the above proof and obtained from the fact that the trace of Q_i equals the sum of its eigenvalues.

3 The 1-homogeneous property

In 1994 Nomura introduced the concept of homogeneous graphs, i.e., the graphs which have the 1-homogeneous property, and established their basic properties, see for example [13, Lemma 1], where he showed that any 1-homogeneous graph is distance-regular. The main result of this section is a characterization of distance-regular graphs with $a_1 \neq 0$ having the CAB property by 1-homogeneous graphs with $a_1 \neq 0$. In the second part of this section we pay special attention to eigenvalues of local graphs of a 1-homogeneous graph with diameter d, $a_1 \neq 0$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. It turns out that $-1 - b_1/(1+\theta)$, for some $\theta \in \{\theta_1, \theta_d\}$, is an eigenvalue of all local graphs. For a distance-regular graph with the CAB property we recursively determine all its intersection numbers in terms of the eigenvalues of a local graph and intersection numbers $c_2, \ldots, c_{d-1}$. We end this section with some parameter restrictions which will be used in the following section.

Recall that if Γ is a distance-regular graph with diameter d, then there exist constants p^i_{jh}, $i, j, h \in \{0, 1, \ldots, d\}$, called **intersection numbers**, such that for any vertices x and y of Γ at distance i we have $|\Gamma_j(x) \cap \Gamma_h(y)| = p^i_{jh}$, and observe $|D^h_j(x,y)| = p^i_{jh}$. If $i = j + h$ then $D^h_j(x,y)$ is nonempty. On the other hand, the set $D^h_j(x,y)$ is empty when $i > j + h$ by the triangle inequality.

Theorem 3.1 *Let Γ be a distance-regular graph with diameter d and $a_1 \neq 0$. Then Γ is 1-homogeneous if and only if it has the CAB property.*

Proof. We may assume $d \geq 2$. Let x and y be adjacent vertices of Γ, and $D^j_h = D^j_h(x,y)$. The assumption $a_1 \neq 0$ implies, by [2, Prop. 5.5.1(i)], $a_i \neq 0$ for $i = 1, \ldots, d-1$. Observe

$$|D^i_{i-1}| = |D^{i-1}_i| = \frac{b_1 b_2 \ldots b_{i-1}}{c_1 c_2 \ldots c_{i-1}}, \qquad |D^i_i| = |\Gamma_i(y)| - |D^i_{i-1}| - |D^i_{i+1}| = a_i \frac{b_1 b_2 \ldots b_{i-1}}{c_1 c_2 \ldots c_i}. \qquad (6)$$

Therefore the sets D^i_{i+1}, D^{i+1}_i and D^i_i are nonempty for $i = 1, \ldots, d-1$.

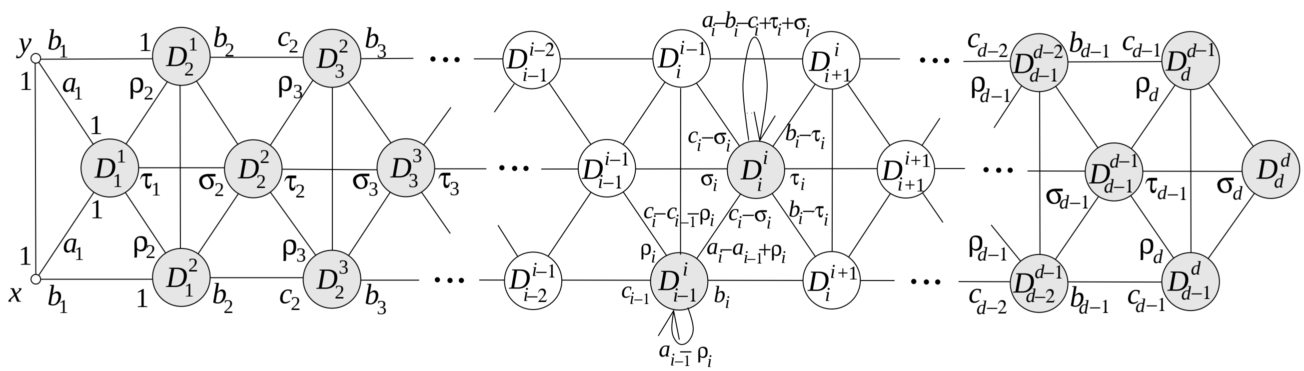

Figure 3.1: The distance partition corresponding to an edge, cf. [12, Lemma 2.11].

Let $i \in \{1, \ldots, d\}$. If $i \neq d$, or $i = d$ and $a_d \neq 0$, then let $z \in D^i_i$, i.e., $x \in A_i(z,y)$. Let $w \in D^i_{i-1}$, i.e., $x \in C_i(w,y)$. Then

$$\alpha_i(x,y,z) := |\Gamma_{i-1}(z) \cap D^1_1|, \qquad \beta_i(x,y,z) := |\Gamma_{i+1}(z) \cap D^1_1| \quad \text{and} \quad \gamma_i(x,y,w) := |\Gamma_{i-1}(w) \cap D^1_1|.$$

Then $\alpha_1 = 1$, $\gamma_1 = 0$, and $\beta_d = 0$. Let us define also

$$\tau_i(x,y,z) := |\Gamma(z) \cap D^{i+1}_{i+1}|, \qquad \sigma_i(x,y,z) := |\Gamma(z) \cap D^{i-1}_{i-1}| \quad \text{and} \quad \rho_i(x,y,w) := |\Gamma(w) \cap D^{i-1}_{i-1}|,$$

see Figure 3.1. Then $\rho_1 = \sigma_1 = 0$, $\tau_d = 0$ when $a_d \neq 0$, and $\tau_{d-1} = 0$ when $a_d = 0$.

Let us first suppose Γ is 1-homogeneous. By a two way counting of geodesics of length $i-1$, $i \neq 1$, between the vertex z (resp. w) and vertices in D_1^1, see Figure 3.1, we obtain

$$\sigma_i \sigma_{i-1} \cdots \sigma_2 = \alpha_i(x,y,z)\, c_{i-1} \cdots c_2 c_1, \tag{7}$$

$$\sum_{j=2}^{i} c_{i-1} \cdots c_j \rho_j \sigma_{j-1} \cdots \sigma_2 = \gamma_i(x,y,w)\, c_{i-1} \cdots c_2 c_1. \tag{8}$$

Similarly, the number of paths of length i, $i < d$, between the vertex z and vertices in D_1^1 equals $\Big(a_1 - \beta_i(x,y,z) - \alpha_i(x,y,z)\Big)\, c_i c_{i-1} \cdots c_1$. On the other hand, this number equals the number of all paths of length i (these are not necessary geodesics, but their number can be determined from Figure 3.1 in terms of the parameters τ_j, ρ_j, σ_j, b_j, c_j, a_j for $j \leq i$ of the 1-homogeneous partition) minus $c_i c_{i-1} \cdots c_1 \sum_{j=1}^{i} a_j$ (i.e., the number of those paths whose ends are at distance $i-1$ from each other). The above relations for $\alpha_i(x,y,z)$, $i = 2, \ldots, d-1$ and $i = d$ when $a_d \neq 0$, for $\beta_i(x,y,z)$, $i = 1, \ldots, d-1$ and for $\gamma_i(x,y,z)$, $i = 2, \ldots, d$ imply that these numbers do not depend on choice of vertices x, y and z, such that $x \in A_i(z,y)$, and $w \in C_i(z,y)$, for each i, therefore, as $\delta_i = \gamma_{i+1}$ for $i = 1, \ldots, d-1$, cf. (1), the graph Γ has the CAB property.

Now we show the converse. Suppose Γ has the CAB property and that we already have shown σ_j, ρ_j and τ_j for $j < i$ are independent of choice of adjacent vertices x and y, and vertices in D_j^j, $D_j^{j-1} \cup D_{j-1}^j$, D_j^j respectively. Then the above two way counting gives us for $i > 1$ and $i \neq d$ when $a_d = 0$ the following relations

$$\sigma_i(x,y,z)\, \sigma_{i-1} \cdots \sigma_2 = \alpha_i c_{i-1} \cdots c_2 c_1, \tag{9}$$

$$\rho_i(x,y,w)\, \sigma_{i-1} \cdots \sigma_2 + \sum_{j=2}^{i-1} c_{i-1} \cdots c_j \rho_j \sigma_{j-1} \cdots \sigma_2 = \gamma_i c_{i-1} \cdots c_2 c_1. \tag{10}$$

Since the RHS of (9) is always nonzero by Theorem 2.4, and the product $\sigma_{i-1} \cdots \sigma_2$ is nonzero as well, the relation (9) implies that $\sigma_i(x,y,z)$ is independent of the vertices x, y and z. Therefore, by induction, this is true for all $\sigma_i(x,y,z)$, $i = 2, \ldots, d-1$ and $i = d$ when $a_d \neq 0$. Similarly, we use induction and (10) to show that $\rho_i(x,y,z)$ is independent of the vertices x, y and w, for $i = 2, \ldots, d$. If Γ is locally disconnected, then, by Theorem 2.4(i), $\gamma_i = 0$, which implies $\rho_i = 0$ for $i = 2, \ldots, d$, and thus $\tau_i = b_i$ for $i = 1, \ldots, d-1$, so we are done. Therefore, we assume from now on that Γ is locally connected.

We now take a closer look at the third relation from the first part of the proof, where we stated the sum $c_i c_{i-1} \cdots c_1 (a_1 - \beta_i - \alpha_i + \sum_{j=1}^{i} a_j)$ is a function of parameters τ_j, ρ_j, σ_j, b_j, c_j, a_j for $j \leq i$. Let us denote it by f_i. In order to prove that $\tau_i(x,y,z)$ is independent of the vertices x, y and z by a similar induction argument as above, it suffices to show that this function is linear in τ_i. The function f_i counts the number of paths of length i from the vertex z to the set D_1^1. Starting in the vertex z such a path can continue to $a_i - b_i - c_i + \tau_i + \sigma_i$ candidates in D_i^i, to $2(c_i - \sigma_i)$ candidates in $D_i^{i-1} \cup D_{i-1}^i$, or to σ_i candidates in D_{i-1}^{i-1}, therefore

$$f_i = (a_i - b_i - c_i + \tau_i(x,y,y) + \sigma_i) z_1 + 2(c_i - \sigma_i) z_2 + \sigma_i f_{i-1},$$

where z_1 and z_2 are the RHS's of (9) and (10) respectively, and $z_1 \neq 0$ as Γ is locally connected. ∎

Remark 3.2 *Let Γ be a distance-regular graph with diameter d, $a_1 \neq 0$ and the CAB property. Let the parameters $\tau_i = \tau_i(x,y,z)$, $\sigma_i = \sigma_i(x,y,z)$, $\rho_i = \rho_i(x,y,w)$ be defined as in the above proof. The two way countings from the proof of Theorem 11.2 in [12, pp. 24-25] give us*

$$\alpha_i c_{i-1} = \sigma_i\, \alpha_{i-1}, \qquad \beta_{i-1} b_i = \tau_{i-1}\, \beta_i, \qquad \gamma_i(c_{i-1} - \sigma_{i-1}) = \rho_i\, \alpha_{i-1} \qquad \textit{for } i = 2, \ldots, d.$$

By Theorem 2.4, we have $\alpha_{i-1} \neq 0$, and if Γ is locally connected, then also $\beta_i \neq 0$. If Γ is locally disconnected, then $\gamma_i = 0$ by Theorem 2.4 implies $\rho_i = 0$ for $i = 2, \ldots, d$, and thus $\tau_i = b_i$ and $\sigma_{i+1} = c_{i+1}a_i/a_{i+1}$ by (6) for $i = 1, \ldots, d-1$, see Figure 3.2.

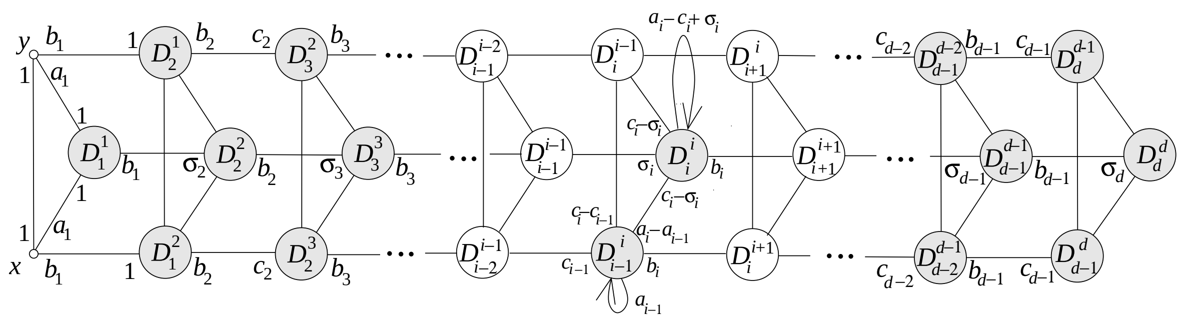

Figure 3.2: The distance partition corresponding to an edge of a locally disconnected 1-homogeneous graph with diameter d and $a_1 \neq 0$.

Therefore, the above relations imply directly that all σ_i's, τ_i's and ρ_i's, but the parameter τ_{d-1} in the case when Γ is locally connected (since $\beta_d = b_d = 0$), are independent of vertices x, y, z and w.

Theorem 3.1 implies that the local property CAB_j, for j small, is weakening of the 1-homogeneous property. The above result and Theorem 2.3 imply a slight generalization of Nomura's result [13, Thm. 1].

Corollary 3.3 *A locally disconnected 1-homogeneous graph with diameter $d \geq 2$ and $a_1 \neq 0$ is a regular near $2d$-gon.* ∎

Remark 3.4 *A distance-regular graph with at most one i such that $a_i \neq 0$ is 1-homogeneous by Nomura [13, Lemma 2], for example, bipartite graphs, generalized Odd graphs and the Wells graph.*

Proposition 3.5 *Any cubic distance-regular graph is 1-homogeneous.*

Proof. By Biggs, Boshier and Shawe-Taylor [1], [2, Thm. 7.5.1], the only cubic distance-regular graphs that are not already 1-homogeneous by Remark 3.4 are the dodecahedron, the Coxeter graph and the Biggs-Smith graph, and it is easy to check they are 1-homogeneous, see Figure 3.3. ∎

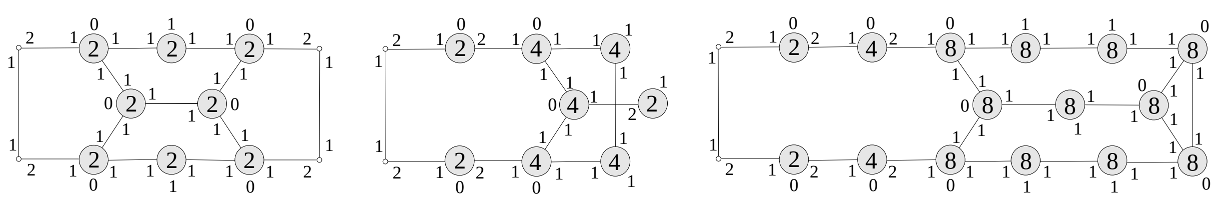

Figure 3.3: The 1-homogeneous partition for the dodecahedron, the Coxeter graph and the Biggs-Smith graph.

3.1 Eigenvalues of a local graph

Let x and y be vertices of a graph Γ. Then we denote the distance of x and y by $\partial_\Gamma(x,y)$ or just by $\partial(x,y)$. Let Γ be a distance-regular graph with diameter $d \geq 2$ and $a_1 \neq 0$. For each pair of adjacent vertices x and y of Γ we define the scalar $f(x,y)$ by

$$f(x,y) := \frac{1}{a_1}|\{(z,w) \in V(\Gamma) \times V(\Gamma) \mid z,w \in D_1^1(x,y),\ \partial(z,w) = 2\}|,$$

Observe $f(x,y)$ is the average degree of the complement of the λ-graph, i.e., the subgraph of Γ induced by the common neighbours of x and y.

Theorem 3.6 (cf. [12, Theorem 3.5]) *Let Γ be a distance-regular graph with diameter $d \geq 2$, $a_1 \neq 0$ and let θ_1 and θ_d be its second largest and the smallest eigenvalue respectively. Let x and y be adjacent vertices of Γ. Then*

$$b_1 \frac{k+\theta_d(a_1+1)}{(k+\theta_d)(1+\theta_d)} \quad \leq \quad f(x,y) \quad \leq \quad b_1 \frac{k+\theta_1(a_1+1)}{(k+\theta_1)(1+\theta_1)}. \tag{11}$$

Let $\theta \in \{\theta_1, \theta_d\}$ and $b := -1 - b_1/(\theta+1)$. If

$$f(x,y) = b_1 \frac{k+\theta(a_1+1)}{(k+\theta)(1+\theta)}, \tag{12}$$

then b is an eigenvalue of the local graph $\Delta(x)$, and if, furthermore, $b = a_1$, then $\Delta(x)$ is disconnected.

Proof. Consider the following matrix

$$L := \begin{pmatrix} 0 & \tilde{b}_0 & 0 \\ \tilde{c}_1 & \tilde{a}_1 & \tilde{b}_1 \\ 0 & \tilde{c}_2 & \tilde{a}_2 \end{pmatrix} = \begin{pmatrix} 0 & a_1 & 0 \\ 1 & a_1 - f(x,y) - 1 & f(x,y) \\ 0 & \dfrac{a_1 f(x,y)}{b_1} & \dfrac{a_1(b_1 - f(x,y))}{b_1} \end{pmatrix}, \tag{13}$$

where the intersection numbers labeled by tilde correspond to the corresponding averages in the partition $\{\{y\}, D_1^1(x,y), D_1^2(x,y)$ of $\Delta(x)\}$. Let α_1 and α_2 be the second largest and smallest eigenvalue of L respectively and let $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_k$ be the eigenvalues of $\Delta(x)$. By interlacing, see Haemers [7], [8], we know that the eigenvalues of L interlace the eigenvalues of $\Delta(x)$. This means

$$\eta_k \leq \alpha_2 \qquad \text{and} \qquad \alpha_1 \leq \eta_2. \tag{14}$$

It follows a_1 is the largest eigenvalue of L. Let $g(x) := (x-\alpha_1)(x-\alpha_2)$, and $b^+ = -1 - b_1/(1+\theta_d)$, $b^- = -1 - b_1/(1+\theta_1)$. By Terwilliger [15], [2, Thm. 4.4.3], we know

$$b^- \leq \eta_k \qquad \text{and} \qquad \eta_2 \leq b^+. \tag{15}$$

and therefore, $g(b^+) \geq 0$ and $g(b^-) \geq 0$. The first (resp. second) one gives us the left (resp. right) inequality of (11). Straightforward calculations show (12) holds if and only if the eigenvalues of L are a_1, b, $a_1\theta/(k+\theta)$. If the left (resp. right) inequality of (11) is satisfied with equality, then $b^+ = \alpha_1$ (resp. $b^- = \alpha_2$) and for $b = b^+$ (resp. $b = b^-$) we have $b \leq \eta_2 \leq b$ by (14) and (15), so b is an eigenvalue of $\Delta(x)$. Finally, if $b = a_1$ and (12) holds, then $\theta = -1 - b_1/(1+a_1)$ and $f(x,y) = 0$, thus the local graph $\Delta(x)$ is disconnected. ∎

Suppose Γ is a distance-regular graph with diameter $d \geq 2$ and $a_1 \neq 0$ whose local graphs are strongly regular. Then the following statement follows immediately from Theorem 3.6 and interlacing, since the quotient matrix Q_1 is equal to the matrix L as defined in (13), and b is the eigenvalue of L if and only if (12) holds.

Corollary 3.7 *Let Γ be a distance-regular with diameter $d \geq 2$, $a_1 \neq 0$ and let θ_1 and θ_d be its second largest and the smallest eigenvalue respectively. Suppose that the local graphs of Γ are strongly regular. Let x and y be adjacent vertices of Γ and let $a_1 = \eta_1 \geq \eta_2 \geq \ldots \geq \eta_k$ be the eigenvalues of the local graph $\Delta(x)$. Then the following are equivalent for $\theta \in \{\theta_1, \theta_d\}$.*

(i) $f(x,y) = b_1 \dfrac{k+\theta(a_1+1)}{(k+\theta)(1+\theta)}$,

(ii) $b := -1 - b_1/(\theta+1)$ *is an eigenvalue of* $\Delta(x)$ *and* $b \leq \eta_2$.

Moreover, suppose (i) and (ii) hold. Then

(iii) if $b = a_1$, then the local graphs of Γ are the disjoint union of (a_1+1)-cliques,

(iv) if Γ is locally connected, then the eigenvalues of the local graph $\Delta(x)$ are a_1, b, $a_1\theta/(k+\theta)$. ∎

Remark 3.8 *The following two examples show that the assumption of Γ being locally strongly regular in Corollary 3.7 is not completely redundant.*

(i) Let Γ be the direct product of a Shrikhande graph, see Figure 3.4, and a 4-clique. Then Γ is a Doob graph, see [4], thus it is distance-regular graph with intersection array $\{9,6,3;1,2,3\}$, the smallest eigenvalue θ_3 is -3, see [2, p. 426], and local graphs that are the disjoint union of a triangle and a hexagon. Let x be a vertex of Γ and let $y \in \Gamma(x)$. If y is a vertex of the triangle in $\Delta(x)$, then $f(x,y) = 0$. Hence (12) holds with $\theta = \theta_3$ and b^+ is an eigenvalue of $\Delta(x)$. Since $b^+ = a_1$, it has multiplicity at least two by Theorem 3.6. On the other hand, if y is vertex of the hexagon, then $f(x,y) \neq 0$, and hence (12) does not hold with $\theta = \theta_3$.

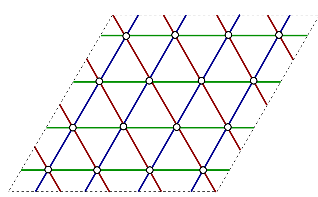

Figure 3.4: The Shrikhande graph drawn on a torus.

(ii) Let Γ be a Johnson graph $J(2n+1, n)$, $n \geq 2$. Then it has diameter $d = n$, $k = n(n+1)$, $a_1 = 2n-1$, the second largest eigenvalue $\theta_1 = n^2 - n - 1$, the smallest eigenvalue $\theta_d = -n$, and the local graphs of Γ are $n \times (n+1)$-grid graphs. It follows that for adjacent vertices x and y of Γ we have $f(x,y) = 2n(n-1)/(2n-1)$, which is not equal to either the upper or lower bound for $f(x,y)$ in (11). On the other hand $b^- = -2$ is an eigenvalue of $\Delta(x)$ by [2, Prop. 1.12.1].

Theorem 3.9 *Let Γ be a distance-regular graph with diameter $d \ge 2$ and $a_1 \ne 0$. Let θ_1, θ_d be respectively the second largest and smallest eigenvalue of Γ. Then (i) implies (ii), where*
(i) Γ *is* 1*-homogeneous;*
(ii) $-1 - b_1/(1+\theta)$ *for some* $\theta \in \{\theta_1, \theta_d\}$ *is an eigenvalue of all local graphs of* Γ.

Proof. By Corollary 3.7 and Proposition 2.1, it suffices to prove that the condition (i) of Corollary 3.7 holds for some $\theta \in \{\theta_1, \theta_d\}$. For adjacent vertices x and y of Γ let us define the vector space $W(x, y)$ by

$$W(x,y) = \mathrm{Span}\{w_{ij}(x,y) \mid 0 \le i, j \le d,\ p^1_{ij} \ne 0\}$$

where $w_{ij}(x, y)$ is the characteristic vector of $D^i_j(x, y)$. We have $\dim(W(x, y)) \le 3d$. $W(x, y) \supseteq H(x, y) := \mathrm{Span}\{w_{01}(x, y), w_{10}(x, y), w_{11}(x, y)\}$. Let Γ be 1-homogeneous. Then, by [12, Lemma 11.5], $AW(x, y) \subseteq W(x, y)$, where A is the adjacency matrix of Γ. Therefore $m := \dim MH(x, y) \le 3d$, where M is the Bose-Mesner algebra of Γ. This means that $t := 3d + 1 - m \ge 1$. Now the statement follows by [12, Definition 5.1, Theorem 5.2 and Corollary 3.6]. ∎

Remark 3.10 *For a distance-regular graph Γ with strongly regular local graphs the condition (ii) of Theorem 3.9 does not imply that Γ is* 1*-homogeneous, as there is a distance-regular graph with intersection array $\{6, 4, 2, 1; 1, 1, 4, 6\}$ which satisfies (ii) with eigenvalue $\theta_4 = -3$, see [2, p.421] and is not* 1*-homogeneous (because it has $a_d = 0$ and is locally disconnected, therefore it is not a tight graph in the sense of [12], and hence not* 1*-homogeneous, see [12, Thm. 11.7]).*

Theorem 3.11 *Let Γ be a locally connected* 1*-homogeneous graph with diameter $d \ge 2$, $a_1 \ne 0$ and let $a_1 > p > q$ be the eigenvalues of a local graph of Γ. Then one of the following holds.*
(i) $a_d = 0$ *and* Γ *is tight in the sense of [12].*
(ii) $a_d \ne 0$ *and* Q_d *has eigenvalue* p.
(iii) $a_d \ne 0$ *and* Q_d *has eigenvalue* q.

In each of (i)-(iii), all the intersection numbers of Γ can be expressed in terms of a_1, p, q, $c_2, c_3, \ldots, c_{d-1}$.

Proof. In view of Theorem 2.7 we only have to show that if $a_d \ne 0$ and we know the smallest eigenvalue ρ of Q_d, then we can express a_d and c_d in terms of γ_d, i.e., δ_{d-1}. But this follows easily by Lemma 2.6 as $a_1 + \rho = \mathrm{tr}(Q_d) = \gamma_d + a_1 - \alpha_d$, $a_d + c_d = k$ and $c_d(a_1 - \gamma_d) = \alpha_d a_d$. ∎

Proposition 3.12 *Let Γ be a* 1*-homogeneous graph with diameter d and $a_1 \ne 0$, where (k, a_1, λ', μ') are the parameters of a local strongly regular graph, and let $i \in \{1, \ldots, d-1\}$. Then $\delta_i \ge \max(\gamma_i, \gamma_i + \mu' - \lambda')$. Furthermore, if Γ contains an induced quadrangle or $\lambda' < \mu'$, then $c_{i+1} > c_i$ and $b_{i-1} > b_i$.*

Proof. Let $i \in \{1, \ldots, d-1\}$, let u, v be vertices of Γ at distance $i + 1$, and let $w \in C_{i+1}(v, u)$. Then $C_i(w, v) \subseteq C_{i+1}(u, v)$ and therefore $c_i \le c_{i+1}$ and $\gamma_i \le \gamma_{i+1} = \delta_i$. The trace $\mathrm{tr}(Q_i)$ is at least $\gamma_i + a_1 - \delta_i$. Since $\mathrm{tr}(Q_i) = a_1 + p + q = a_1 + \lambda' - \mu'$, this means $\delta_i \ge \gamma_i - \lambda' + \mu'$.

If $\lambda' < \mu'$, then $\gamma_{i+1} > \gamma_i$, and this implies $c_{i+1} > c_i$. So we may assume Γ contains an induced quadrangle. Suppose $c_i = c_{i+1}$. Then $C_{i+1}(u, v) = C_i(w, v)$ and thus also $\gamma_{i+1} = \gamma_i$, i.e., $\delta_i = \delta_{i-1}$. Now Theorem 2.7 implies $b_{i+1} = b_i$ and hence $c_{i+1} - c_i = b_{i+1} - b_i$, which is in contradiction with [2, Theorem 5.2.1]. The proof of $b_{i-1} > b_i$ is similar using Remark 2.8(i). ∎

4 Terwilliger graphs

In the introduction we defined Terwilliger graphs in the case when they are distance-regular. There is also a more general definition and it is not difficult to see that it coincides with the above one in the case of distance-regular graphs. Let Γ be a graph and x, y be vertices of Γ at distance two. Then the $\mu(x,y)$-graph is the subgraph of Γ induced by the common neighbours of x and y. A connected graph with diameter at least two is called a **Terwilliger graph** when every μ-graph has the same number of vertices and is complete, see for example [2, p. 34] or [14]. In this section we study a Terwilliger graph Γ that satisfies the following conditions:

(A) the intersection numbers k, a_1, c_2 and a_2 exist, i.e., for a vertex x of Γ the partition $\{x\} \cup \Gamma(x) \cup \Gamma_2(x)$ is equitable and its parameters are independent of x,

(B) $a_2 \neq 0$ and there exists a constant $\alpha \geq 1$ such that for all vertices x, y of Γ, at distance two, and all $z \in D_2^1(x,y)$ we have
$$|\Gamma(z) \cap D_1^1(x,y)| = \alpha.$$

The condition (A) is obviously satisfied in the case Γ is distance-regular and the condition (B) is satisfied when Γ with $a_2 \neq 0$ has additionally the CAB$_2$ property, therefore also for 1-homogeneous graphs with $a_2 \neq 0$, since $\alpha = \alpha_2$, $\alpha_2 \neq 0$ by Theorem 2.4; cf. Remark 3.2 and $\alpha = |\Gamma(z) \cap \Gamma(y) \cap \Gamma(x)| = |D_1^1(y,z) \cap \Gamma(x)| = \sigma_2$. Since $|D_1^1(x,y)| = c_2$ we have $\alpha \leq c_2$. By counting the edges between the sets $D_1^1(x,y)$ and $D_2^1(x,y)$ and noting that the set $D_1^1(x,y)$ induces a complete graph we find

$$\alpha a_2 = c_2(a_1 - c_2 + 1) \tag{16}$$

We prove that $\alpha \in \{c_2 - 1, c_2\}$ and condider these two cases seperately. We show $\alpha = c_2$ if and only if $c_2 = 1$. In the case $\alpha = c_2 - 1$ we show all the local graphs of Γ are strongly regular. Furthermore, we show that $c_2 \geq 2$ implies $c_2 = 2$, $\alpha = 1$ and that all local graphs of Γ are Moore graphs with diameter two. Then we show a Terwilliger graph with $c_2 \geq 2$ satisfying the condition (A) has CAB$_2$ property if and only if it satisfies the condition (B). Now we give a finite algorithm for determining all possible intersection arrays of 1-homogeneous Terwilliger graphs, whose local graphs are strongly regular with given parameters. We apply this algorithm to classify 1-homogeneous graphs whose local graphs are Moore graphs with diameter two. Therefore, we obtain a classification of all 1-homogeneous Terwilliger graphs.

Lemma 4.1 *If Γ is a Terwilliger graph, satisfying the conditions (A) and (B), then $\alpha \in \{c_2 - 1, c_2\}$, i.e., $c_2 - 1 \leq \alpha \leq c_2$.*

Proof. Let x and y be vertices of Γ at distance two and let $z \in D_2^1(x,y)$. Define $A := \Gamma(z) \cap D_1^1(x,y)$ and suppose $\alpha \leq c_2 - 2$. Then there are vertices $u, v \in D_1^1(x,y) \backslash A$ and the subgraph induced by $\{u\} \cup \{v\} \cup A$ is complete, since the set $D_1^1(x,y)$ induces a complete graph by the definition of a Terwilliger graph. So $\partial(u,v) = 1$, $\partial(u,z) = 2 = \partial(v,z)$ and the set $\Gamma(u) \cap D_1^1(v,z)$ contains $A \cup \{y\}$, which means that $\alpha = |A| \geq |A \cup \{y\}| = \alpha + 1$ by the property (B). Contradiction! Hence $\alpha \geq c_2 - 1$. ∎

Lemma 4.2 *Let Γ be a Terwilliger graph satisfying the conditions (A) and (B). Then $\alpha = c_2$ if and only if $c_2 = 1$.*

Proof. Let $\alpha = c_2$ and let x, y be vertices of Γ at distance two. If there existed two distinct nonadjacent vertices u and v of $D_2^1(x,y)$ then we would have $D_1^1(u,v) \supseteq D_1^1(x,y) \cup \{y\}$, and hence $c_2 \geq c_2 + 1$, which is not possible. Therefore, the set $D_2^1(x,y)$ induces a clique. If $d = 2$, then $D_3^1(x,y) = \emptyset$, $\Delta(y)$ is a clique and therefore Γ is complete. Assume now that $d \geq 3$. Let $z_1 \in D_1^1(x,y)$, $z_2 \in D_3^1(x,y)$. Then $D_1^1(z_1,z_2) \subseteq D_2^1(x,y) \cup \{y\}$ by the definition of a Terwilliger graph, and thus $c_2 \leq a_2 + 1$. This implies that if $a_2 = 0$, then $c_2 = 1$. Furthermore, if $a_2 = 1$ and $c_2 \geq 2$, then $\alpha = c_2 = 2$ and, by (16), $a_1 = 2$, which means that there are no edges between $D_2^1(x,y)$ and $D_3^1(x,y)$. So y is the only common neighbour of z_1 and z_2 and hence $c_2 = 1$. We may now assume $a_2 \geq 2$. Since the set $D_2^1(x,y)$ induces a clique, a vertex $D_2^1(x,y)$ has $a_1 - \alpha - (a_2 - 1)$ neighbours in $D_3^1(x,y)$. But by (16) and $\alpha = c_2$, this number is zero, and $D_1^1(z_1,z_2) = \{y\}$, hence $c_2 = 1$.

For the converse we assume $c_2 = 1$. Then, by $0 < \alpha \leq c_2$, $\alpha = 1 = c_2$. ∎

1-Homogeneous Terwilliger graphs with $a_1 \neq 0$ and $c_2 = 1$ are regular $2d$-gons. Let us point out that a local graph of a Terwilliger graph is again a Terwilliger graph.

Proposition 4.3 *Let Γ be a Terwilliger graph satisfying the conditions (A), (B) with $\alpha = c_2 - 1$. Then all local graphs of Γ are strongly regular.*

Proof. Since $1 \leq \alpha = c_2 - 1$, we have $c_2 \geq 2$. Let y be a vertex of Γ and let us define a relation on $\Gamma(y)$ by

$$z \equiv u \iff \{z\} \cup D_1^1(z,y) = \{u\} \cup D_1^1(u,y).$$

Then this is an equivalence relation. By Terwilliger [2, Thm. 1.16.3], [14], the quotient corresponding to the equivalence classes of the above relation is strongly regular, and there exists a number ℓ such that all the equivalence classes have size ℓ and $\ell | c_2 - 1$. We will show $\ell | c_2$ and hence $\ell = 1$, which implies the local graph $\Delta(y)$ is strongly regular.

Take a vertex $x \in \Gamma_2(y)$ and let $u \in D_1^1(x,y)$. Suppose $v \in \Delta(y)$ and $v \equiv u$. Since $D_1^1(x,y)$ is a clique which contains u, the vertex v is adjacent to all the vertices in $D_1^1(x,y) \backslash \{v\}$. But $\alpha \leq c_2 - 1$, so $v \in D_1^1(x,y)$. Therefore, the set $D_1^1(x,y)$ is a union of equivalence classes and $\ell | c_2$. ∎

A **Moore graph** of diameter two is a regular graph with girth five and diameter two.

Theorem 4.4 *Let Γ be a Terwilliger graph satisfying the conditions (A) and (B) with $c_2 \geq 2$. Then $c_2 = 2$, $\alpha = 1$ and local graphs of Γ are Moore graphs with diameter two.*

Proof. Since $c_2 \geq 2$, we have $\alpha = c_2 - 1$ by Lemmata 4.1 and 4.2. Suppose $c_2 \geq 3$ and let y be a vertex of Γ. Then, by Proposition 4.3, the local graph $\Delta(y)$ is a strongly regular Terwilliger graph with $\mu' = c_2 - 1 \geq 2$ satisfying the condition (B) with $\alpha' = \alpha - 1 \geq 1$. Let x and z be vertices of $\Gamma(y)$ at distance two. Let Ω be the subgraph of Γ induced by $D_1^1(x,y)$. Then it is strongly regular by Proposition 4.3 applied to $\Delta(y)$. The partition $\{D_1^1(x,z)\backslash\{y\}, D_1^1(x,y)\backslash D_1^1(x,z)\}$ (the first set is nonempty, because $c_2 \geq 3$, and the second, because $\alpha \geq 2$ implies $a_1 - \mu' \neq 0$) of Ω is equitable with the quotient matrix

$$\begin{pmatrix} c_2 - 2 & a_1' - c_2 + 2 \\ c_2 - 2 & a_1' - c_2 + 2 \end{pmatrix},$$

where a_1' is the valency of Ω. But this means that zero is an eigenvalue of Ω, which is not possible as Ω is strongly regular and a Terwilliger graph. Therefore $c_2 = 2$ and $\alpha = 1$. By Proposition 4.3, it follows $\Delta(x)$ is a Moore graph of diameter two, as $\Delta(x)$ does not have triangles and $\mu' = 1$. ∎

Although we have defined the CAB$_j$ property only for distance-regular graphs, we only really need the existance of parameters a_i, b_i and c_i for $i \leq j$.

Corollary 4.5 *Let Γ be a Terwilliger graph with $c_2 \geq 2$, satisfying the condition (A). Then the following are equivalent.*

(i) Γ *satisfies the CAB$_2$ property,*

(ii) Γ *satisfies the condition (B).*

Moreover, if (i) and (ii) hold, then Γ is locally a Moore graph with diameter two.

Proof. (i) $\Rightarrow$ (ii): Follows immediately from $\alpha = \alpha_2$.
(ii) $\Rightarrow$ (i): This follows from $\alpha_2 = \alpha = 1$, $c_2 = 2$, the fact that local graphs of Γ are Moore graphs of diameter two, and the fact that Moore graphs are 1-homogeneous. ∎

Remark 4.6 *The only Moore graphs of diameter two are the pentagon, the Petersen graph, the Hoffman-Singleton graph and possibly a strongly regular graph on 3250 vertices. No locally Hoffman-Singleton graph is known although there are feasible parameters, for example* $\{50, 42, 9; 1, 2, 42\}$ *and* $\{50, 42, 1; 1, 2, 50\}$*, cf. [2, p.36].*

We give an algorithm to calculate all possible intersection arrays of 1-homogeneous graphs for which we know that local graphs are connected strongly regular graphs with given parameters, see Algorithm 4.7. Given the parameters of a connected strongly regular graph, this algorithm is finite for 1-homogeneous graphs whose local graphs are strongly regular with these parameters, since the inner for loop assumes $c_i \geq c_{i-1} + 1$. Let us explain why we can do this.

By Corollary 4.5, the local graphs of 1-homogeneous Terwilliger graphs with $a_1 \neq 0$ and $c_2 \geq 2$ are Moore graphs, and hence have $\lambda' = 0 < 1 = \mu'$. Finally, by Proposition 3.12, any 1-homogeneous graph that contains an induced quadrangle or for which $\lambda' < \mu'$, satisfies $c_i > c_{i-1}$ for $i = 2, \ldots, d$.

The set F contains all possible intersection arrays, for which we then need to check feasibility.

Algorithm 4.7 *Given the parameters (k', λ', μ') of a connected strongly regular graph, calculate its eigenvalues $k' = a_1 > p > q$ and the parameters*

$$k = v' = \frac{(a_1 - p)(a_1 - q)}{a_1 + pq}, \qquad b_1 = k - a_1 - 1, \qquad \alpha_1 = 1, \qquad \beta_1 = a_1 - \lambda' - 1, \qquad \gamma_1 = 0, \qquad \delta_1 = \mu'.$$

and initialize the sets $F := \emptyset$ (final), $N := \emptyset$ (new) and $S := \{\{k, b_1, \delta_1\}\}$ (current).

for $i \geq 2$ **and** $S \neq \emptyset$ **do**

  for $\{c_2, \ldots, c_{i-1}, \delta_{i-1}; k, b_1, \ldots, b_{i-1}\} \in S$ **do**

    $\gamma_i := \delta_{i-1}$;

    if $\gamma_i = a_1$ **then** $a_i = 0; c_i = k$; $F := F \cup \{\{k, b_1, \ldots, b_{i-1}; 1, c_2, c_3, \ldots, c_i\}\}$ **fi**;

    if $\gamma_i < a_1$ **then**

      assume diameter $= i$ *and calculate* α_i, a_i, c_i *(by Theorem 3.11);*

      if $\left(k_i \in \mathbb{N} \;\; \textbf{and} \;\; \alpha_i,\ a_i,\ c_i \in \mathbb{N} \;\; \textbf{and} \;\; a_i(a_1 - \alpha_i)/2,\ c_i\gamma_i/2 \in \mathbb{N}_0\right)$

      then $F := F \cup \{\{k, b_1, \ldots, b_{i-1}; 1, c_2, \ldots, c_i\}\}$ **fi**;

      assume diameter $> i$;

      for $c_i = \max(c_{i-1}, \gamma_i) + 1, \ldots, b_1$ **do**

        calculate $\alpha_i, \beta_i, \delta_i, b_i, a_i$ *(by Theorem 2.7);*

        if $\left(k_i \in \mathbb{N} \;\; \textbf{and} \;\; \alpha_i,\ \beta_i,\ \delta_i,\ b_i,\ a_i \in \mathbb{N} \;\; \textbf{and} \;\; \delta_i \geq \gamma_i \;\; \textbf{and} \;\; \frac{c_i\gamma_i}{2}, \frac{(a_1 - \beta_i - \alpha_i)a_i}{2}, \frac{b_i(a_1 - \delta_i)}{2} \in \mathbb{N}_0\right)$

        then $N := N \cup \{\{c_2, \ldots, c_i, \delta_i; k, b_1, \ldots, b_i\}\}$ **fi**;

      od;

    fi;

  od;

  $S := N; \quad N := \emptyset$;

od;

Remark 4.8 *Although the above algorithm will give in the case of Moore graphs all possible feasible intersection arrays, there is a more direct approach using the following relation for calculating α_i, β_i, δ_i, a_i, b_i, c_i for $i \in \{2, \ldots, d-1\}$. Let $i \in \{2, \ldots, d-1\}$. We have $\gamma_i(\gamma_i + 1) = a_1 - 1$ if and only if $a_1 = \delta_i + \gamma_i$. If $\gamma_i(\gamma_i + 1) \neq a_1 - 1$, then*

$$\beta_i = a_1 - \delta_i + \gamma_i + 1 + \frac{\gamma_i(\gamma_i + 1) - a_1 + 1}{a_1 - \delta_i - \gamma_i}.$$

This relation is obtained in a similar way as the relations of Theorem 2.7, and gives a nontrivial divisibility condition. We know γ_i and the above relation gives us a finite number of possiblities for δ_i, since β_i is a positive integer. From δ_i and γ_i we then calculate the parameters β_i, α_i, c_i, a_i, b_i.

Theorem 4.9 *A graph whose local graphs are Moore graphs is* 1*-homogeneous if and only if it is one of the following graphs:*

(i) *the icosahedron with intersection array* $\{5, 2, 1; 1, 2, 5\}$,

(ii) *the Doro graph with intersection array* $\{10, 6, 4; 1, 2, 5\}$*, see [2, Prop. 12.2.2],*

(iii) *the Conway-Smith graph with intersection array* $\{10, 6, 4, 1; 1, 2, 6, 10\}$*, see [2, Sec. 13.2.B].*

(iv) *the complement of the triangular graph* $T(7)$ *with intersection array* $\{10, 6; 1, 6\}$*, see [2, Sec. 12.1].*

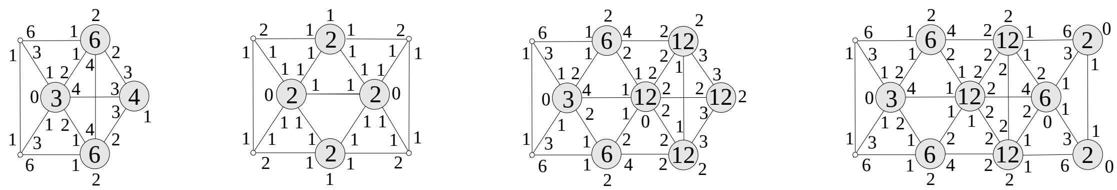

Figure 4.1: The 1-homogeneous partition for the complement of the triangular graph $T(7)$, the icosahedron, the Doro graph and the Conway-Smith graph.

Proof. The only graph that is locally pentagon is the icosahedron, see [2, Prop. 1.1.4]. There are three locally Petersen graphs, all three distance-regular, see Hall [9], [2, Thm. 1.16.5]. The icosahedron and the Conway-Smith graph are tight and therefore 1-homogeneous by [12, Sec. 13]. The complement of the triangular graph $T(7)$ is the antipodal quotient of the Conway-Smith graph and it is thus 1-homogeneous. We obtain that the Doro graph is 1-homogeneous by a direct verification of the CAB_3 property.

For a 1-homogeneous graph whose local graphs are the Hoffman-Singleton graph with intersection array $\{7, 6; 1, 1\}$ the following feasible arrays are obtained by Algorithm 4.7 and Remark 4.8:

$$
\begin{aligned}
d = 2: &\quad \{50, 42; 1, 20\},\\
d = 3: &\quad \{50, 42, 36; 1, 2, 25\},\\
d = 4: &\quad \{50, 42, 36, 5(5-i); 1, 2, 5i, 30\}, \quad \text{for } i = 1, 2, 3, 4,\\
d = 5: &\quad \{50, 42, 36, 5(5-i), 2; 1, 2, 5i, 36, 40\}, \quad \text{for } i = 1, 2, 3, 4,\\
d = 6: &\quad \{50, 42, 36, 5, 2, 1; 1, 2, 20, 36, 42, 50\},
\end{aligned}
$$

but none of them has integral eigenvalue multiplicities.

For a 1-homogeneous graph whose local graphs are a Moore graphs on 3250 vertices and with intersection array $\{57, 56; 1, 1\}$, the following feasible arrays are obtained:

$$
\begin{aligned}
d = 3: &\quad \{3250, 3192, 3136; 1, 2, 500\},\\
d = 4: &\quad \{3250, 3192, 3136, 550; 1, 2, 225, 2470\},\\
d = 4: &\quad \{3250, 3192, 3136, 550; 1, 2, 225, 2650\},\\
d = 4: &\quad \{3250, 3192, 3136, 1540; 1, 2, 60, 1495\},\\
d = 4: &\quad \{3250, 3192, 3136, 1540; 1, 2, 60, 1900\},\\
d = 4: &\quad \{3250, 3192, 3136, 2970; 1, 2, 5, 650\},\\
d = 5: &\quad \{3250, 3192, 3136, 2970, 1144; 1, 2, 5, 78, 2080\},\\
d = 5: &\quad \{3250, 3192, 3136, 2970, 1144; 1, 2, 5, 78, 2350\},
\end{aligned}
$$

but again none of them has integral eigenvalue multiplicities. ∎

The following result is a direct consequence of Theorem 4.9, and the fact that only the first three graphs Theorem 4.9 are Terwilliger graphs, see Figure 4.1.

Corollary 4.10 *A Terwilliger graph Γ with $c_2 \geq 2$ is 1-homogeneous if and only if Γ is one of the following graphs:*

(i) the icosahedron with intersection array $\{5,2,1;1,2,5\}$,

(ii) the Doro graph with intersection array $\{10,6,4;1,2,5\}$, see [2, Prop. 12.2.2],

(iii) the Conway-Smith graph with intersection array $\{10,6,4,1;1,2,6,10\}$, see [2, Sec. 13.2.B].

5 Conclusion

Suppose Γ is a distance-regular graph with diameter $d \geq 3$ and the CAB$_2$ property. If the local graphs of Γ are the conference graphs, then Algorithm 4.7 implies $b_2 = 1$ and therefore it must be double-cover of a complete graph, i.e., Taylor graph, by Jurišić and Koolen [10, Thm. 2.2]. Since the conference graphs are the only strongly regular graphs that could have irrational eigenvalues, this implies that in the case when $d \geq 4$ the eigenvalues of the local graphs are integral.

In this paper we classified the Terwilliger 1-homogeneous graphs with $a_1 \geq 1$ and $c_2 \geq 2$. This implies that together with [2, Thm. 5.2.1] that for an 1-homogeneous graph whose local graph are connected strongly regular graph with parameters (k, a_1, λ', μ') we may assume that

$$b_i - c_i \geq b_{i-1} - c_{i-1} + a_1 + 2, \qquad \text{for } i = 2, \ldots, d.$$

It seems that for 1-homogeneous graphs with $a_1 = 0$ we cannot say too much. A local approach as is done in this paper is not possible in this case, because all distance-regular graphs with $a_1 = 0$ have the CAB property.

We classified the Terwillger 1-homogeneous graphs with $a_1 \neq 0$ and $c_2 \geq 2$ in the paper, i.e., all μ-graphs are cliques. It would be interesting to classify all 1-homogeneous graphs whose μ-graphs are complete multipartite. In a follow up paper we do this for the family of tight nonbipartite antipodal distance-regular graphs with diameter four.

Finally, we propose the following two problems.

Question 1: Is a Terwilliger distance-regular graph 1-homogeneous?

Question 2: Let Γ be a 1-homogeneous locally connected distance-regular graph with diameter d, $a_d \neq 0$, and let a local graph have nontrivial eigenvalues p and q, $p > q$. Does the following hold: $p = -1 - b_1/(\theta_d + 1)$ if and only if q is an eigenvalue of Q_d?

ACKNOWLEDGEMENT: The idea to study the CAB$_j$ property was inspired by the list of problems of Paul Terwilliger [16], and we would like to thank him for giving us this list.

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6 Things to do or some Open problems

1. Instead of 1-homogeneous graphs study distance-regular graphs with diameter d and the CAB$_{d-1}$ property.

2. Classify distance-regular graphs with diameter $d \geq 3$, the CAB$_2$ property, $\alpha_2 = 1$ or 2, and whose μ-graphs are complete bipartite graphs
   (triangular extension when EGQ is a graph, see Johnatan Hall and Shult)

   Start with $GQ(s,t)$ containing regular points and try to construct 1-homogeneous graph.

3. Classify distance-regular graphs with diameter $d \geq 3$, the CAB$_2$ property, and whose μ-graphs are the complete multipartite graphs $K_{t \times n}$.