IX. Tight distance-regular graphs

- alternative proof of the fundamental bound
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- characterizations
- examples
- parametrization
- AT4 family
- complete multipartite μ-graphs
- classifications of AT4(qs, q, q) family
- uniqueness of the Patterson graph
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Lemma. *Let $\Gamma =$ be a k-regular, connected graph on n vertices, e edges and t triangles, with eigenvalues*

$$k = \eta_1 \leq \eta_2 \leq \cdots \leq \eta_n.$$

Then

(i) $\sum_{i=1}^{n} \eta_i = 0,$

(ii) $\sum_{i=1}^{n} \eta_i^2 = nk = 2e,$

(iii) $\sum_{i=1}^{n} \eta_i^3 = nk\lambda = 6t,$

if λ is the number of triangles on every edge.

Now suppose that r and s are resp. an upper and lower bounds on the nontrivial eigenvalues.
Hence $(\eta_i - r)(\eta_i - s) \leq 0$ for $i \neq 1$, and so

$$\sum_{i=2}^{k} (\eta_i - s)(\eta_i - r) \leq 0,$$

which is equivalent to

$$n(k + rs) \leq (k - s)(k - r).$$

Equality holds if and only if

$$\eta_i \in \{r, s\} \text{ for } i = 2, \ldots, n,$$

i.e., Γ is strongly regular with eigenvalues k, r and s.

Let us rewrite this for a local graph of a vertex of a distance-regular graph:

$$k \leq \frac{(a_1 - b^-)(a_1 - b^+)}{a_1 + b^- b^+}.$$

where b^- and b^+ are the lower and the upper bound for the nontrivial eigenvalues of the local graph.

We define for a distance-regular graph with diam. d and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$

$$b^- = -1 - \frac{b_1}{\theta_1 + 1} \quad \text{and} \quad b^+ = -1 - \frac{b_1}{\theta_d + 1},$$

and note $b^- < 0$ and $b^+ > 0$.

Theorem [Terwilliger]. *Let x be a vertex of a distance-regular graph Γ with diameter $d \geq 3$, $a_1 \neq 0$ and let*

$$a_1 = \eta_1 \geq \eta_2 \geq \ldots \geq \eta_k$$

be the eigenvalues of the local graph $\Delta(x)$. Then,

$$b^+ \geq \eta_2 \geq \eta_k \geq b^-.$$

Proof. Let us define N_1 to be the adjacency matrix of the local graph $\Delta = \Delta(x)$ for the vertex x and let N to be the Gram matrix of the normalized representations of all the vertices in Δ.

Since Γ is not complete multipartite, we have $\omega_2 \neq 1$ and

$$\begin{aligned} N &= I_k + N_1\omega_1 + (J_k - I_k - N_1)\omega_2 \\ &= (1-\omega_2)\left(I_k + N_1\frac{\omega_1-\omega_2}{1-\omega_2} + J_k\frac{\omega_2}{1-\omega_2}\right). \end{aligned}$$

The matrix $N/(1-\omega_2)$ is positive semi-definite, so its eigenvalues are nonegative and we have for $i = 2, \ldots, k$:

$$1 + \frac{\omega_1-\omega_2}{1-\omega_2}\eta_i \geq 0, \quad \text{i.e.,} \quad 1 + \frac{1+\theta}{\theta+b_1+1}\eta_i \geq 0.$$

Since k is the spectral radious, by the expression for $1-\omega_2$, we have $\theta > -b_1 - 1$ and thus also

$$(1+\theta)\eta_i \geq -(\theta + b_1 + 1).$$

If $\theta > -1$, then

$$\eta_i \geq -\frac{\theta + b_1 + 1}{\theta + 1} = -1 - \frac{b_1}{\theta + 1}.$$

The expression on the RHS is an increasing function, so it is uper-bounded by b^-.

Similarly if $\theta < -1$, then η_i is lower-bounded by b^+.
■

Fundamental bound (FB) [JKT'00]

Γ distance-regular, diam. $d \geq 2$,
and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$.

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \geq \frac{-ka_1b_1}{(a_1+1)^2}$$

If equality holds in the FB and Γ is nonbipartite, then Γ is called a **tight graph**.

For $d=2$ we have $b_1 = -(1+\theta_1)(1+\theta_2)$, $b^+ = \theta_1$, $b^- = \theta_2$, and thus **Γ is tight** (i.e., $\theta_1 = 0$) **iff $\Gamma = K_{t\times n}$ with $t > 2$** (i.e., $a_1 \neq 0$ and $\mu = k$).

Characterizations of tight graphs

Theorem [JKT'00]. *A nonbipartite distance-regular graph Γ with diam. $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. TFAE*

(i) Γ *is tight.*

(ii) Γ *is 1-homogeneous and* $a_d = 0$.

(iii) the local graphs of Γ *are connected strongly regular graphs with eigenvalues* a_1, b^+, b^-, *where*

$$b^- = -1 - \frac{b_1}{\theta_1 + 1} \quad \text{and} \quad b^+ = -1 - \frac{b_1}{\theta_d + 1}.$$

Examples of tight graphs

- the Johnson graph $J(2d, d)$,
- the halved cube $H(2d, 2)$,
- the Taylor graphs,

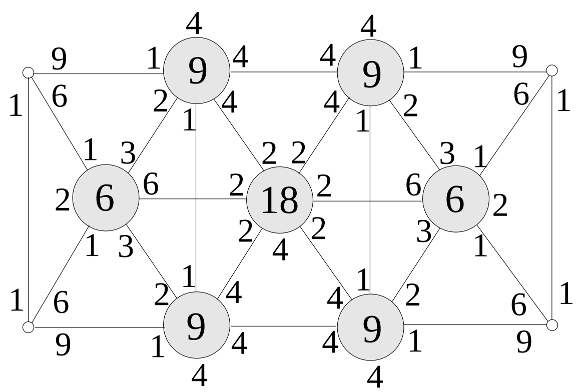
- the AT4 family
  (antipodal tight DRG with diam. 4),
- the Patterson graph $\{280, 243, 144, 10; 1, 8, 90, 280\}$
  (related to the sporadic simple group of Suzuki).

(i) The **Johnson graph $J(2d, d)$** has diameter d and intersection numbers

$$a_i = 2i(d-i), \quad b_i = (d-i)^2, \quad c_i = i^2 \quad (i = 0, \ldots, d).$$

It is distance-transitive, antipodal double-cover and Q-polynomial with respect to θ_1.

Each local graph is a **lattice graph $K_d \times K_d$**, with parameters $(d^2, 2(d-1), d-2, 2)$ and nontrivial eigenvalues $r = d-2$, $s = -2$.

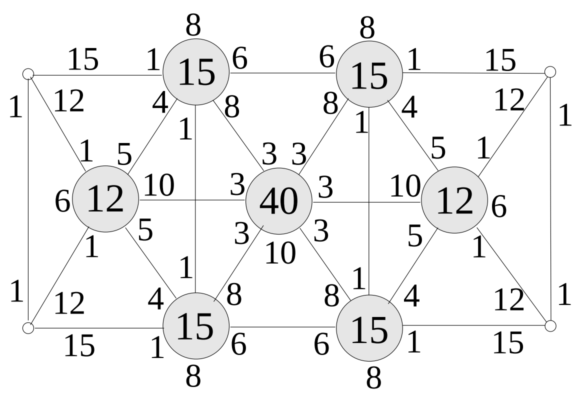


(ii) The **halved cube** $\boldsymbol{H(2d,2)}$ has diameter d and intersection numbers $(i = 0, \ldots d)$

$$a_i = 4i(d-i),\ \ b_i = (d-i)(2d-2i-1),\ \ c_i = i(2i-1).$$

It is distance-transitive, antipodal double-cover and Q-polynomial with respect to θ_1.

Each local graph is a **Johnson graph** $\boldsymbol{J(2d,2)}$, with parameters $(d(2d-1), 4(d-1), 2(d-1), 4)$ and nontrivial eigenvalues $r = 2d-4,\ s = -2$.



(iii) The **Taylor graphs** are the double-covers of complete graphs, i.e., distance-regular graphs with intersection arrays $\{k, c_2, 1; 1, c_2, k\}$. They have diameter 3, and are Q-polynomial with respect to both θ_1, θ_d, given by $\theta_1 = \alpha$, $\theta_d = \beta$, where

$$\alpha + \beta = k - 2c_2 - 1, \quad \alpha\beta = -k, \quad \text{and} \quad \alpha > \beta.$$

Each local graph is strongly-regular with parameters (k, a_1, λ, μ), where $a_1 = k - c_2 - 1$,

$$\lambda = \frac{3a_1 - k - 1}{2}, \; \mu = \frac{a_1}{2}, \; r = \frac{\alpha - 1}{2} \text{ and } s = \frac{\beta - 1}{2}.$$

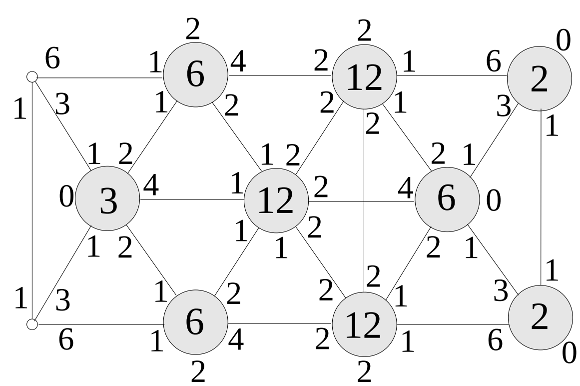
We note both a_1, c_2 are even and k is odd.

For example, the local graphs of the double-cover of K_{18} with $c_2 = 8$ are the **Paley graphs $\boldsymbol{P(17)}$**.

(iv) The **Conway-Smith graph, 3.$Sym(7)$** has intersection array $\{10, 6, 4, 1; 1, 2, 6, 10\}$ and can be obtained from a sporadic Fisher group.

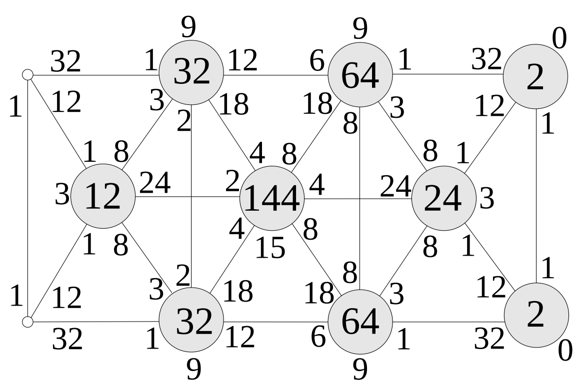
It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is a **Petersen graph**, with parameters $(10, 3, 0, 1)$ and nontrivial eigenvalues $r = 1,\ s = -2.$



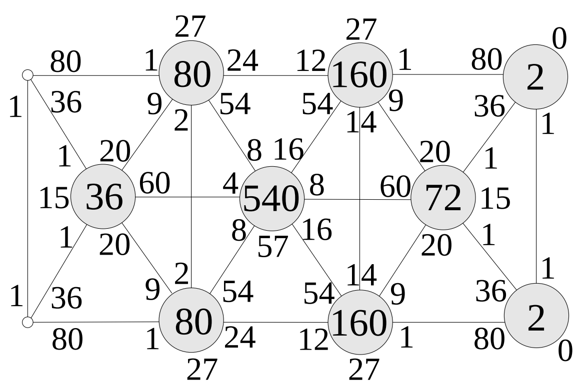
(v) The **$3.O_6^-(3)$-graph** has intersection array $\{45, 32, 12, 1; 1, 6, 32, 45\}$ and can be obtained from a sporadic Fisher group. It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is a **generalized quadrangle** $\boldsymbol{GQ(4,2)}$, with parameters $(45, 12, 3, 3)$ and nontrivial eigenvalues $r = 3$, $s = -3$.



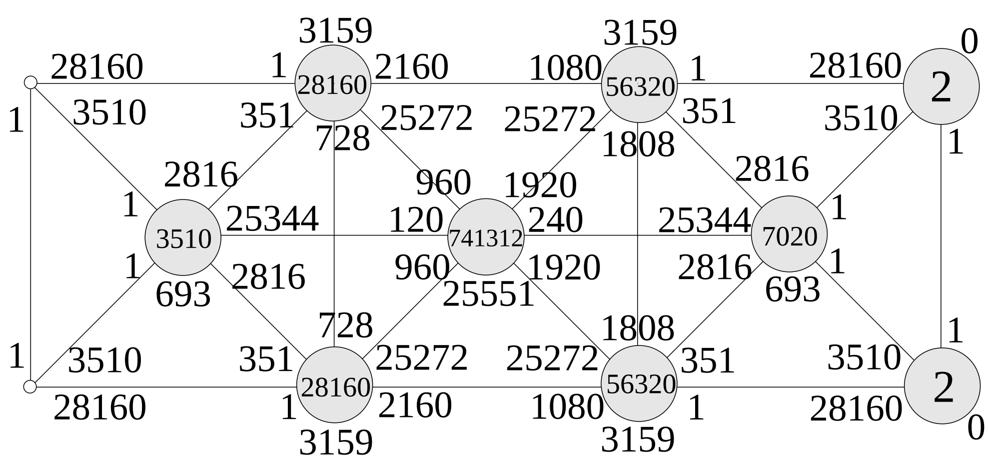
(vi) The **3.O_7(3)-graph** has intersection array $\{117, 80, 24, 1; 1, 12, 80, 117\}$ and can be obtained from a sporadic Fisher group. It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is strongly-regular with parameters $(117, 36, 15, 9)$, and nontrivial eigenvalues $r = 9$, $s = -3$.



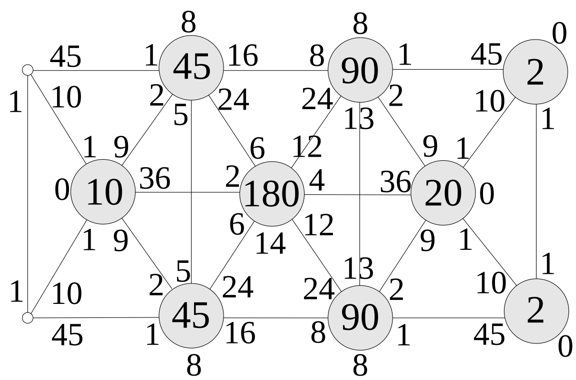
(vii) The **$3.Fi_{24}^{-}$-graph** has intersection array $\{31671, 28160, 2160, 1; 1, 1080, 28160, 31671\}$ and can be obtained from Fisher groups.
It is distance-transitive, antipodal 3-cover and is not Q-polynomial.

Each local graph is **SRG(31671,3510,693,351)** and $r = 351$, $s = -9$. They are related to Fi_{23}.



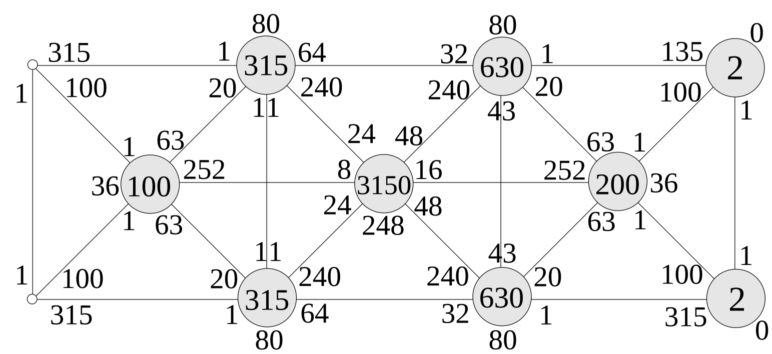
(viii) The **Soicher1 graph** has intersection array $\{56, 45, 16, 1; 1, 8, 45, 56\}$. It is antipodal 3-cover and is not Q-polynomial.

Each local graph is the **Gewirtz graph** with parameters $(56, 10, 0, 2)$ and $r = 2$, $s = -4$.



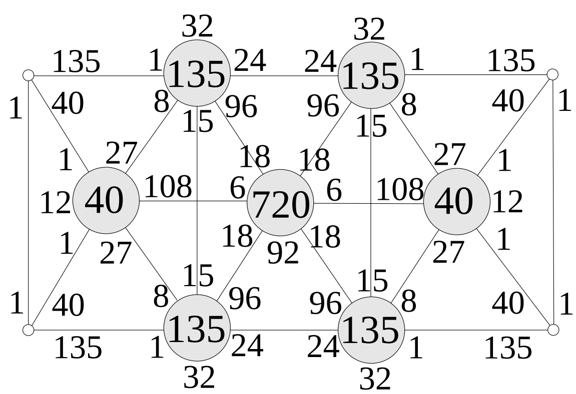
(ix) The **Soicher2 graph** has intersection array $\{416, 315, 64, 1; 1, 32, 315, 416\}$. It is antipodal 3-cover and is not Q-polynomial.

Each local graph is **SRG(416,100,36,20)** and $r = 20$, $s = -4$.



(x) The **Meixner1 graph** has intersection array $\{176, 135, 24, 1; 1, 24, 135, 176\}$. It is antipodal 2-cover and is Q-polynomial.

Each local graph is **SRG(176,40,12,8)** and $r = 8$, $s = -4$.



(xi) The **Meixner2 graph** has intersection array $\{176, 135, 36, 1; 1, 12, 135, 176\}$. It is antipodal 4-cover and is distance-transitive.

Each local graph is **SRG(176,40,12,8)** and $r = 8$, $s = -4$.