For $\theta \in \mathrm{ev}(\Gamma)$ and associated primitive idempotent E:

$$E = \frac{m_\theta}{|V\Gamma|} \sum_{h=0}^{d} \omega_h A_h \quad (0 \le i \le d),$$

$\omega_0, \ldots, \omega_d$ is the **cosine sequence** of E (or θ).

Lemma. *Γ distance-regular, diam. $d \ge 2$, E is a primitive idempotent of Γ corresponding to θ, $\omega_0, \ldots, \omega_d$ is the cosine sequence of θ.*
For $x, y \in V\Gamma$, $i = \partial(x, y)$ we have
(i) $\langle Ex, Ey \rangle =$ *xy-entry of* $E = \omega_i \dfrac{m_\theta}{|V\Gamma|}$.
(ii) $\omega_0 = 1$ *and* $c_i\omega_{i-1} + a_i\omega_i + b_i\omega_{i+1} = \theta\omega_i$ *for* $0 \le i \le d$.

$$\omega_1 = \frac{\theta}{k}, \qquad \omega_2 = \frac{\theta^2 - a_1\theta - k}{kb_1}$$

and

$$\omega_1 - \omega_2 = \frac{(k-\theta)(a_1+1)}{kb_1}, \quad 1 - \omega_2 = \frac{(k-\theta)(\theta+b_1+1)}{kb_1}.$$

Using the Sturm's theorem for the sequence

$$b_0 \dots b_i \, \omega_i(x)$$

we obtain

Theorem. *Let $\theta_0 \geq \cdots \geq \theta_d$ be the eigenvalues of a distance regular graph. The sequence of cosines corresponding to the i-th eigenvalue θ_i has precisely i sign changes.*

Modules

Γ distance-regular, diam. $d \geq 2$.
Suppose $a_1 \neq 0$. Then for $i \neq d$, $a_i \neq 0$, i.e., $D_i^i \neq \emptyset$.
Moreover, $D_d^d = \emptyset$ iff $a_d = 0$.

Let w_{ij} be a characteristic vector of the set D_i^j and
$W = W(x,y) := \mathrm{Span}\{w_{ij} \mid i,j = 0,\ldots,d\}$. Then

$$\dim W \;=\; \begin{cases} 3d & \text{if } \; a_d \neq 0, \\ 3d-1 & \text{if } \; a_d = 0. \end{cases}$$

For $\forall xy \in E\Gamma$, we define the scalar $f = f(x,y)$:

$$f = \frac{1}{a_1}\Big|\{(z,w) \in X^2 \mid z,w \in \Gamma(x,y),\ \partial(z,w) = 2\}\Big|.$$

f is the average degree of the complement of the λ-graph. Then $0 \le f \le a_1 - 1, b_1$ and for $\theta \in \mathrm{ev}(\Gamma)$, $E = E(\theta)$ the Gram matrix of $E\hat{x}$, $E\hat{y}$, w_{11} is

$$\frac{m_\theta^3}{n} \det \begin{pmatrix} \omega_0 & \omega_1 & a_1\omega_1 \\ \omega_1 & \omega_0 & a_1\omega_1 \\ a_1\omega_1 & a_1\omega_1 & c \end{pmatrix}$$

where $c = a_1\big(\omega_0 + (a_1 - f - 1)\omega_1 + f\omega_2\big)$.

So

$$(\omega - \omega_2)(1+\omega)f \le (1-\omega)(a_1\omega + 1 + \omega),$$

i.e.,

$$\boxed{(k+\theta)(1+\theta)\ f \ \le \ b_1\big(k+\theta(a_1+1)\big).}$$

We now consider which of $\theta_1, \theta_2, \ldots, \theta_d$ gives the best bounds for f. Let θ denote one of $\theta_1, \theta_2, \ldots, \theta_d$, and assume $\theta \ne -1$. If $\theta > -1$ (resp. $\theta < -1$), the obtained inequality gives an upper (resp. lower) bound for f.

Consider the partial fraction decompostion

$$b_1 \frac{k+\theta(a_1+1)}{(k+\theta)(1+\theta)} \quad = \quad \frac{b_1}{k-1}\left(\frac{ka_1}{k+\theta} + \frac{b_1}{1+\theta}\right).$$

Since the map $F : \mathbb{R} \setminus \{-k, -1\} \longrightarrow \mathbb{R}$, defined by

$$x \mapsto \frac{ka_1}{k+x} + \frac{b_1}{1+x}$$

is strictly decreasing on the intervals $(-k, -1)$ and $(-1, \infty)$, we find that the least upper bound for f is obtained at $\theta = \theta_1$, and and the greatest lower bound is obtained at $\theta = \theta_d$:

$$\boxed{b_1 \frac{k+\theta_d(a_1{+}1)}{(k{+}\theta_d)(1{+}\theta_d)} \le f \le b_1 \frac{k+\theta_1(a_1{+}1)}{(k{+}\theta_1)(1{+}\theta_1)}.}$$

Set $H = H(x,y) := \mathrm{Span}\{\hat{x}, \hat{y}, w_{11}\}$

Suppose Γ **is 1-homogeneous**. So $AW = W$. The Bose-Mesner algebra $\mathcal{M}$ is generated by A, so also $\mathcal{M}W = W = \mathcal{M}H$ ($:= \mathrm{Span}\{mh \,|\, m \in \mathcal{M}, h \in H\}$).

$E_0, E_1, \ldots, E_d$ is a basis for $\mathcal{M}$, so $E_iE_j = \delta_{ij}E_i$ and

$$\mathcal{M}H = \sum_{i=0}^{d} E_iH \qquad \text{(direct sum)},$$

Note $\dim(E_0H) = 1$ and $3 \geq \dim(E_iH) \geq 2$, and $\dim(E_iH) = 2$ implies $i \in \{1, d\}$.

If $t := |\{i \,|\, \dim(E_iH) = 2\}|$, then $t \in \{0,1,2\}$ and $\dim(\mathcal{M}H) = 3d + 1 - t$. Hence $t = 2$ when $a_d = 0$.