VIII. 1-homogeneous graphs

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- examples
- a local approach and the CAB property
- recursive relations on parameters
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- modules

Homogeneous property

(in the sense of Nomura)

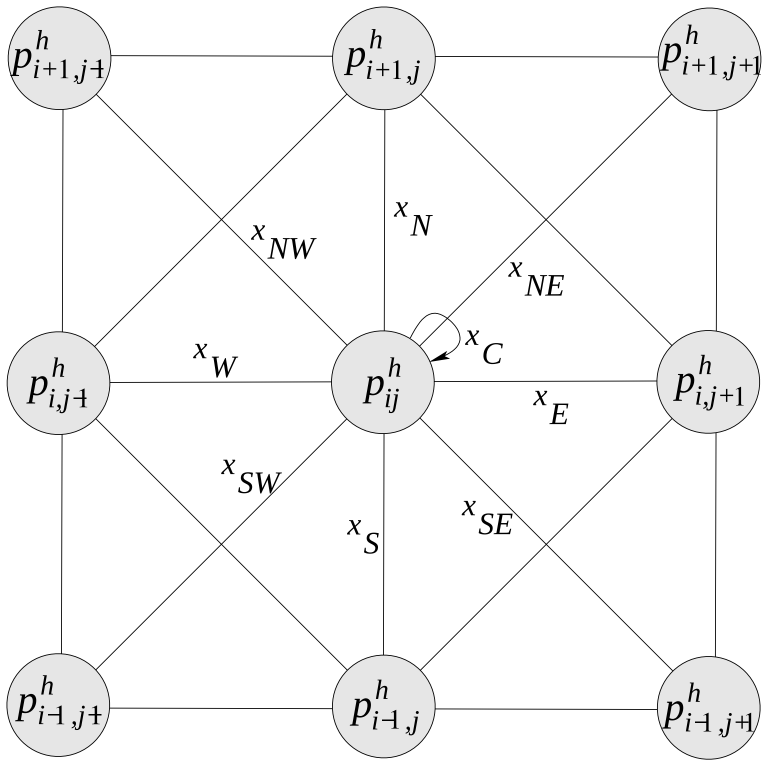
Γ graph, diameter d, $x, y \in V(\Gamma)$, s.t. $\partial(x,y) = h$, $i, j \in \{0, \ldots, d\}$. Set $\boldsymbol{D_i^j = D_i^j(x,y)} := \Gamma_i(x) \cap \Gamma_j(y)$ and note $|D_i^j| = p_{ij}^h$.

The graph Γ is ***h*-homogeneous** when the partition

$$\{D_i^j \mid 0 \le i, j \le d,\ D_i^j \ne \emptyset\}$$

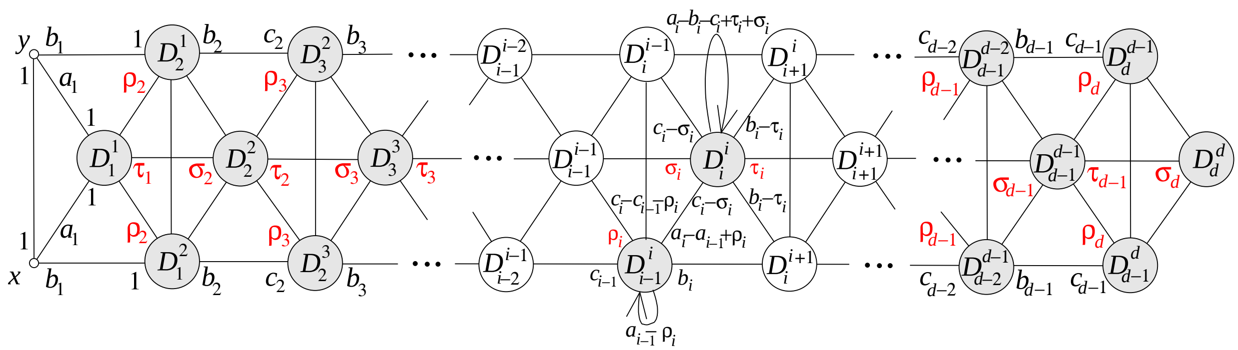
is *equitable* for every $x, y \in V(\Gamma)$, $\partial(x,y) = h$, and the parameters corresponding to equitable partitions are *independent* of x and y.

0-homogeneous $\iff$ distance-regular



$$x_{SW}+x_S+x_{SE}=c_i,\ x_W+x_C+x_E=a_i,\ x_{NW}+x_N+x_{NE}=b_i,$$

$$x_{NW}+x_W+x_{SW}=c_j,\ x_N+x_C+x_S=a_j,\ x_{NE}+x_E+x_{SE}=b_j.$$



For $i \in \{1, \ldots, d\}$

$$|D^i_{i-1}| = |D^{i-1}_i| = \frac{b_1 b_2 \ldots b_{i-1}}{c_1 c_2 \ldots c_{i-1}}, \quad |D^i_i| = a_i \frac{b_1 b_2 \ldots b_{i-1}}{c_1 c_2 \ldots c_i},$$

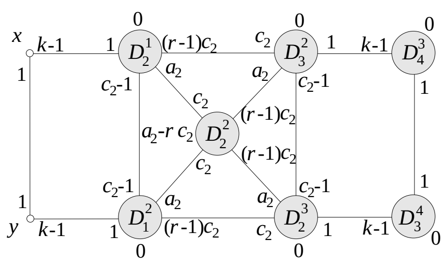
and therefore $D^{i-1}_i \neq \emptyset \neq D^i_{i-1}$.

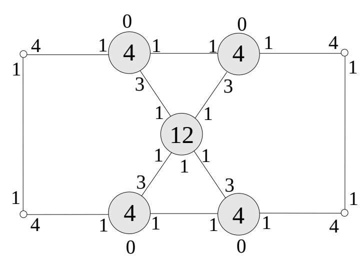
A distance-regular graph Γ is **1-homogeneous** when the distance distribution corresponding to an edge is equitable.

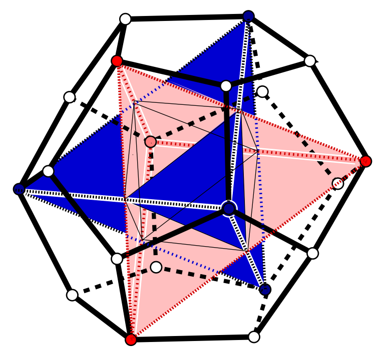
Some examples of 1-homogeneous graphs

distance-regular graphs with at most one i, s.t. $a_i \neq 0$:

– bipartite graphs,
– generalized Odd graphs;

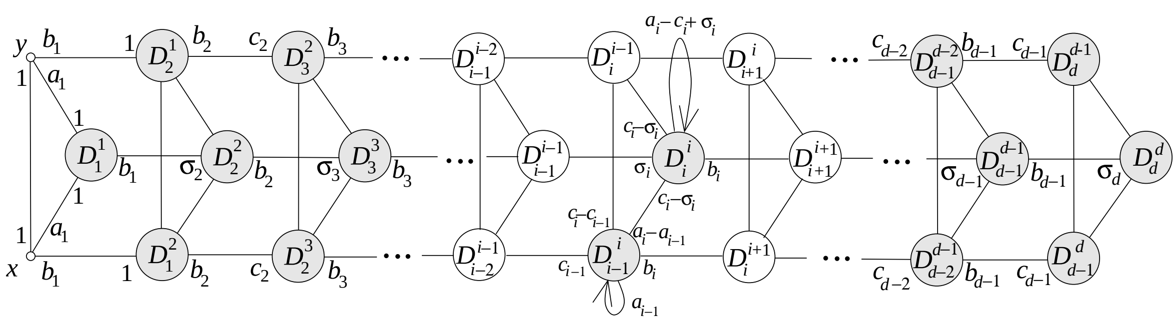






the Wells graph

A 1-homogeneous graph Γ of diameter $d \geq 2$ and $a_1 \neq 0$ is locally disconnected iff it is a regular near $2d$-gon (i.e., a distance-regular graph with $a_i = c_i a_1$ and no induced $K_{1,2,1}$).

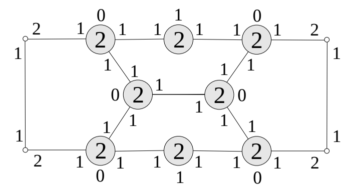


If Γ is locally disconnected, then for $i = 1, \ldots, d-1$.

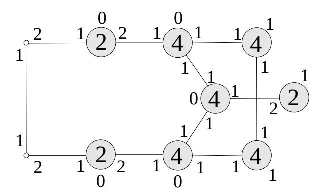
$$\tau_i = b_i \ \text{ and } \ \sigma_{i+1} = \frac{c_{i+1} a_i}{a_{i+1}}.$$

Some examples of 1-homo. graphs, cont.

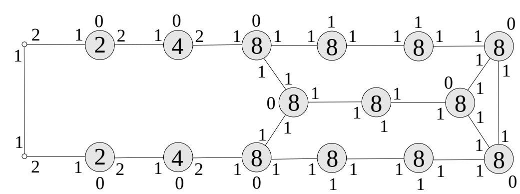
- the Taylor graphs,
- the Johnson graph $J(2d, d)$,
- the folded Johnson graph $\overline{J}(4d, 2d)$,
- the halved n-cube $H(n, 2)$,
- the folded halved $(2n)$-cube,
- cubic distance-regular graphs.



the dodecahedron



the Coxeter graph



the Biggs-Smith graph

The **local graph $\Delta(x)$** is the subgraph of Γ induced by the neighbours of x. It has k vertices & valency a_1.

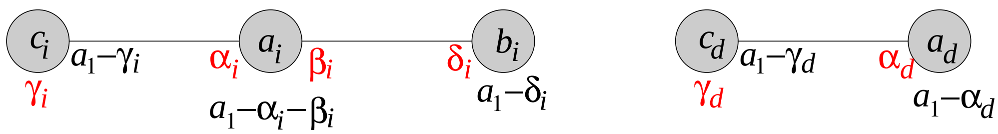
All local graphs of a **1-homogeneous** graph are

(i) connected strongly regular graphs with the same parameters, or

(ii) disjoint unions of (a_1+1)-cliques.

A local approach

For $x, y \in V(\Gamma)$, s.t. $\partial(x,y) = i$, let $\mathbf{CAB}_i(x,y)$ be the partition $\{C_i(x,y), A_i(x,y), B_i(x,y)\}$ of $\Gamma(y)$.



Γ has the $\mathbf{CAB}_j$ **property**, if $\forall i \le j$ the partition $\mathrm{CAB}_i(x,y)$ is equitable $\forall x, y \in V(\Gamma)$, s.t. $\partial(x,y) = i$.

the CAB_1 property $\Longleftrightarrow$ Γ is locally strongly regular

Theorem [JK'00]. Γ *drg, diam.* d, $a_1 \neq 0$. *Then* Γ *is* 1*-homogeneous* $\Longleftrightarrow$ Γ *has the CAB property.*

A two way counting gives us for $i = 2, \ldots, d$:

$$\alpha_i c_{i-1} = \sigma_i \, \alpha_{i-1},$$

$$\beta_{i-1} b_i = \tau_{i-1} \, \beta_i,$$

$$\gamma_i (c_{i-1} - \sigma_{i-1}) = \rho_i \, \alpha_{i-1}.$$

The quotient matrices corresponding to CAB_i partitions are, for $1 \leq i \leq j$, $i \neq d$,

$$Q_i = \begin{pmatrix} \gamma_i & a_1 - \gamma_i & 0 \\ \alpha_i & a_1 - \beta_i - \alpha_i & \beta_i \\ 0 & \delta_i & a_1 - \delta_i \end{pmatrix},$$

and when $j = d$ also $Q_d = \begin{pmatrix} \gamma_d & a_1 - \gamma_d \\ \alpha_d & a_1 - \alpha_d \end{pmatrix}$,

if $a_d \neq 0$, and $Q_d = (\gamma_d)$, if $a_d = 0$

Let Γ be a 1-homogeneous graph with diameter d that is locally connected and let $\delta_0 := 0$.

Then $a_i \neq 0$, $a_1 - \gamma_i \neq 0$, and we have the following recursion: $\gamma_i = \delta_{i-1}$,

$$\alpha_i = \frac{(a_1 - \delta_{i-1})c_i}{a_i}, \qquad \delta_i = \frac{a_i \mu'}{a_1 - \delta_{i-1}}, \qquad \beta_i = b_i \delta_i / a_i,$$

for $i \in \{1, 2, \ldots, d-1\}$, and when $i = d$
$\gamma_d = \delta_{d-1}$, $\alpha_d = (a_1 - \delta_{d-1})c_d/a_d$, if $a_d \neq 0$,
and $\gamma_d = a_1$, if $a_d = 0$.

An bf algorithm to calculate all possible intersection arrays of 1-homogeneous graphs for which we know that local graphs are connected SRGs with given parameters,

Given the parameters (k', λ', μ') of a connected SRG, calculate its eigenvalues $k' = a_1 > p > q$ and

$$k = v' = \frac{(a_1 - p)(a_1 - q)}{a_1 + pq}, \quad b_1 = k - a_1 - 1, \quad \alpha_1 = 1,$$

$$\beta_1 = a_1 - \lambda' - 1, \qquad \gamma_1 = 0, \qquad \delta_1 = \mu'.$$

and initialize the sets $F := \emptyset$ (final), $N := \emptyset$ (new) and $S := \{\{k, b_1, \delta_1\}\}$ (current).

for $i \geq 2$ **and** $S \neq \emptyset$ **do**

for $\{c_2, \ldots, c_{i-1}, \delta_{i-1}; k, b_1, \ldots, b_{i-1}\} \in S$ **do**

$\gamma_i := \delta_{i-1};$

if $\gamma_i = a_1$ **then** $a_i = 0; c_i = k;$ $F := F \cup \{\{k, b_1, \ldots, b_{i-1}; 1, c_2, c_3, \ldots, c_i\}\}$ **fi**;

if $\gamma_i < a_1$ **then**
assume diameter $= i$ and calculate α_i, a_i, c_i

if $\Big(k_i \in \mathbb{N}$ **and** $\alpha_i,\ a_i,\ c_i \in \mathbb{N}$ **and** $a_i(a_1 - \alpha_i)/2,\ c_i\gamma_i/2 \in \mathbb{N}_0\Big)$
then $F := F \cup \{\{k, b_1, \ldots, b_{i-1}; 1, c_2, \ldots, c_i\}\}$ **fi**;

assume diameter $> i$;

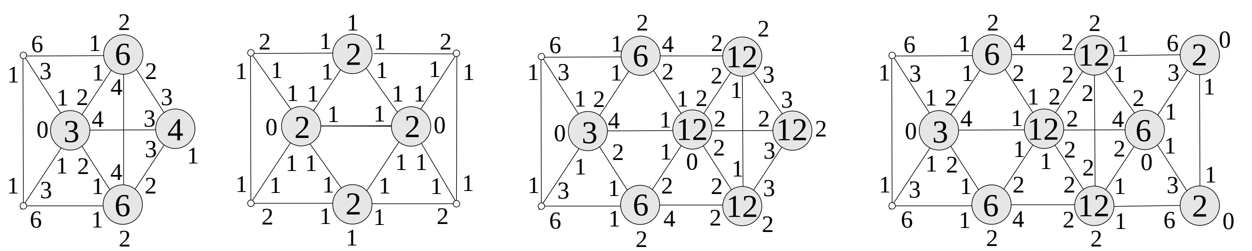
for $c_i = \max(c_{i-1}, \gamma_i) + 1, \ldots, b_1$ **do**
calculate $\alpha_i, \beta_i, \delta_i, b_i, a_i$

if $\Big(k_i \in \mathbb{N}$ **and** $\alpha_i,\ \beta_i,\ \delta_i,\ b_i,\ a_i \in \mathbb{N}$ **and** $\delta_i \geq \gamma_i$
and $\dfrac{c_i\gamma_i}{2},\ \dfrac{(a_1 - \beta_i - \alpha_i)a_i}{2},\ \dfrac{b_i(a_1 - \delta_i)}{2} \in \mathbb{N}_0\Big)$
then $N := N \cup \{\{c_2, \ldots, c_i, \delta_i; k, b_1, \ldots, b_i\}\}$ **fi**;
od;
fi;
od;
$S := N;\quad N := \emptyset;$
od;

Locally Moore graphs

Theorem [JK'00]. *A graph whose local graphs are Moore graphs is* 1*-homogeneous iff it is one of the following graphs:*

- *the icosahedron* ($\{5,2,1;1,2,5\}$),
- *the Doro graph* ($\{10,6,4;1,2,5\}$),
- *the Conway-Smith graph* ($\{10,6,4,1;1,2,6,10\}$),
- *the compl. of* $T(7)$ ($\{10,6;1,6\}$).



Terwilliger graphs

A connected graph with diameter at least two is called a **Terwilliger graph** when every μ-graph has the same number of vertices and is complete.

A distance-regular graph with diameter $d \geq 2$ is a Terwilliger graph iff it contains no induced C_4.

Corollary [JK'00]. *A Terwilliger graph with* $c_2 \geq 2$ *is* 1*-homogeneous iff it is one of the following graphs:*
(i) the icosahedron,
(ii) the Doro graph,
(iii) the Conway-Smith graph.