Theorem. *Let Γ be a distance regular graph and H a distance regular antipodal r-cover of Γ. Then every eigenvalue θ of Γ is also an eigenvalue of H with the same multiplicity.*

PROOF. Let H has diameter D, and Γ has n vertices, so $H_D = n \cdot K_r$ (K_r's corresp. to the fibres of H).

Therefore, H_D has for eigenvalues $r-1$ with multiplicity n and -1 with multiplicity $nr-n$.

The eigenvectors corresponding to eigenvalue $r-1$ are constant on fibres and those corresponding to -1 sum to zero on fibres.

Take θ to be an eigenvalue of H, which is also an eigenvalue of Γ.

An eigenvector of Γ corresponding to θ can be extended to an eigenvector of H which is constant on fibres.

We know that the eigenvectors of H are also the eigenvectors of H_D, therefore, we have $v_D(\theta) = r - 1$.

So we conclude that all the eigenvectors of H corresponding to θ are constant on fibres and therefore give rise to eigenvectors of Γ corresponding to θ. ■

All the eigenvalues: $A(\Gamma/\pi), N_0$ or $A(\Gamma/\pi), N_1$:

$$\begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ 0 & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & \cdot & \cdot & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix}, \begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-2} & a_{d-2} & b_{d-2} \\ & & & & c_{d-1} & a_{d-1} \end{pmatrix}$$

$$\begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ 0 & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & \cdot & \cdot & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix}, \begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d - rt \end{pmatrix}$$

Theorem. *H distance-regular antipodal r-cover, diameter D, of the distance-regular graph Γ, diameter d and parameters a_i, b_i, c_i.*

The $D-d$ eigenvalues of H which are not in $\mathrm{ev}(\Gamma)$ (the 'new' ones) are for $D=2d$ (resp. $D=2d+1$), the eigenvalues of the matrix N_0 (resp. N_1).

If $\theta_0 \geq \theta_1 \geq \cdots \geq \theta_D$ are the eigenvalues of H and $\xi_0 \geq \xi_1 \geq \cdots \geq \xi_d$ are the eigenvalues of Γ, then

$$\xi_0 = \theta_0, \quad \xi_1 = \theta_2, \quad \cdots, \quad \xi_d = \theta_{2d},$$

i.e., the $\mathrm{ev}(\Gamma)$ interlace the 'new' eigenvalues of H.

Connections

– **projective and affine planes**,
for $D = 3$, or $D = 4$ and $r = k$ (covers of K_n or $K_{n,n}$),

– **Two graphs** (Q-polynomial), for $D = 3$ and $r = 2$,

– **Moore graphs**, for $D = 3$ and $r = k$,

– **Hadamard matrices**, $D = 4$ and $r = 2$ (covers of $K_{n,n}$),

– **group divisible resolvable designs**, $D = 4$ (cover of $K_{n,n}$),

– coding theory (perfect codes),
– group theory (class. of finite simple groups),
– orthogonal polynomials.

Tools:

– graph theory, counting,

– matrix theory (rank mod p),

– eigenvalue techniques,

– representation theory of graphs,

– geometry (Euclidean and finite),

– algebra and association schemes,

– topology (covers and universal objects).

Goals:

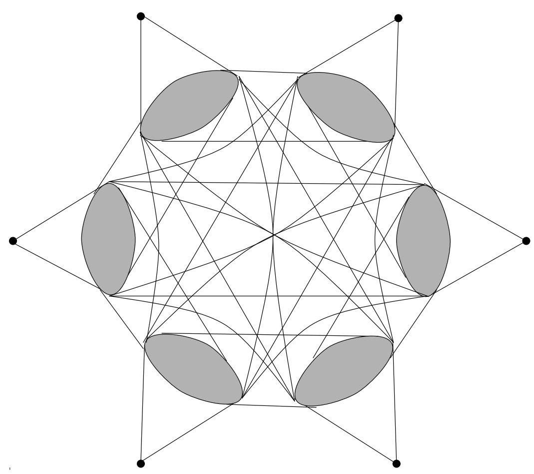
– structure of antipodal covers,
– new infinite families,
– nonexistence and uniqueness,
– characterization,
– new techniques
(which can be applied to drg or even more general)

Difficult problems:

Find a 7-cover of K_{15}.
Find a double-cover of Higman-Sims graph ($\{22, 21; 1, 6\}$).

Antipodal covers of diameter 3

Γ an antipodal distance-regular with diameter 3.
Then it is an r-cover of the complete graph K_n.
Its intersection array is $\{n-1, (r-1)c_2, 1; 1, c_2, n-1\}$.



The distance partition corresp. to an antipodal class.

Examples: 3-cube, the icosahedron.

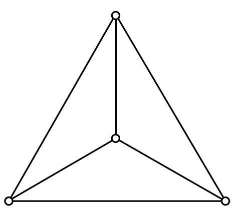
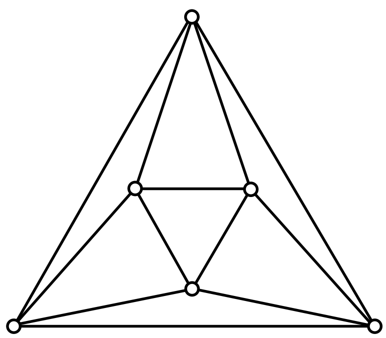
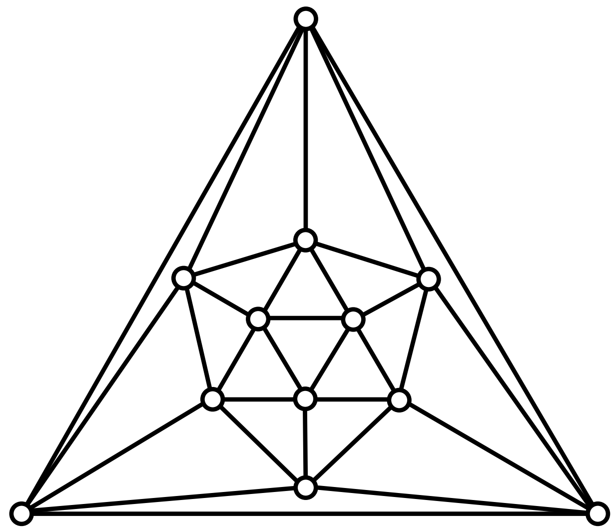
A graph is **locally** $\mathcal{C}$ if the neighbours of each vertex induce $\mathcal{C}$ (or a member of $\mathcal{C}$).

Lemma (A.J. 1994). Γ *distance-regular,* $k \leq 10$ *and locally* C_k. *Then* Γ *is*
– *one of the Platonic solids with* $\triangle$*'s as faces,*
– *Paley graph* $P(13)$, *Shrikhande graph,*
– *Klein graph (i.e., the* 3*-cover of* K_8*).*

Problem. Find a locally C_{15} distance-regular graph.

Platonic solids with △'s as faces

The 1-skeletons of

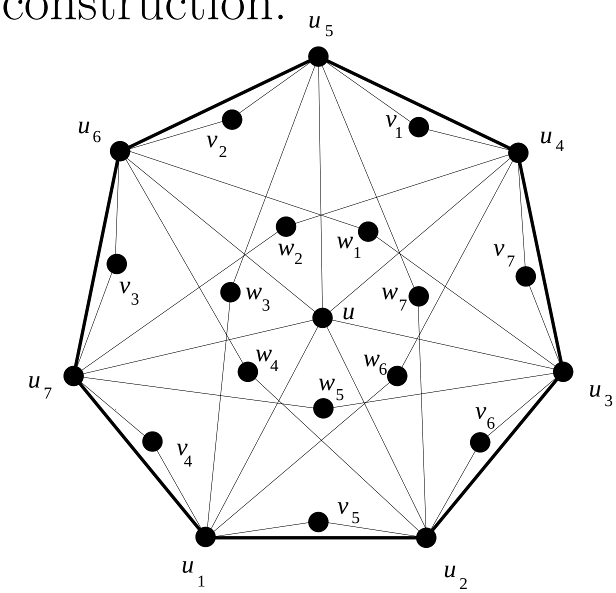




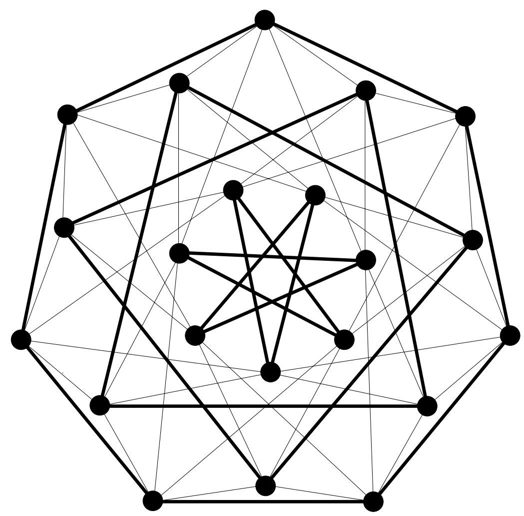
(a) the tetrahedron= K_4,

(b) the octahedron= $K_{2,2,2}$,

(c) the icosahedron.

There is only one feasible intersection array of distance-regular covers of K_8: $\{7,4,1;1,2,7\}$ - the Klein graph, i.e., the dual of the famous Klein map on a surface of genus 3. It must be the one coming from Mathon's construction.





Mathon's construction of an r-cover of K_{q+1}

A version due to Neumaier: using a subgroup K of the $GF(q)^*$ of index r. For, let $q = rc + 1$ be a prime power and either c is even or $q - 1$ is a power of 2.

We use an equivalence relation $\mathcal{R}$ for $GF(q)^2 \backslash \{0\}$: $(v_1, v_2)\mathcal{R}(u_1, u_2)$ iff $\exists h \in K$ s.t. $(v_1 h, v_2 h) = (u_1, u_2)$.

vertices: equiv. classes vK, $v \in GF(q)^2 \backslash \{0\}$ of $\mathcal{R}$, and $(v_1, v_2)K \sim (u_1, u_2)K$ iff $v_1 u_2 - v_2 u_1 \in K$,

It is an antipodal distance-regular graph of diam. 3, with $r(q+1) = (q^2 - 1)/c$ vertices, index r, $c_2 = c$ (vertex transitive, and also distance-transitive when r is prime and the char. of $GF(q)$ is primitive mod r).

Theorem (Brouwer, 1983).
GQ(s,t) *minus a spread,* $t>1$
$\Longrightarrow$ $(s+1)$-*cover of* K_{st+1} *with* $c_2 = t-1$.

- good construction: q a prime power:

$$(s,t) = \begin{cases} (q,q), \\ (q-1,q+1), \\ (q+1,q-1), \text{ if } 2 \mid q \\ (q,q^2). \end{cases}$$

- good characterization (geometric graphs),
- nonexistence

n	r	a_1	c_2	a cover Γ of K_n	#of Γ
5	3	1	1	L(Petersen)	1
6	2	2	2	Icosahedron	1
7	6	0	1	S_2(Hoffman-Singleton)	1
8	3	2	2	Klein graph	1
9	3	1	3	$GQ(2,4)\setminus$ spread	2
9	7	1	1	equivalent to the unique $PG(2,8)$	1
10	2	4	4	Johnson graph $J(6,3)$	1
10	4	2	2	$GQ(3,3)\setminus$ unique spread	≥ 1

n	r	a_1	c_2	a cover Γ of K_n	# of Γ
11	9	1	1	does not exist (PG(2,10))	0
12	5	2	2	Mathon's construction	≥ 1
13	11	1	1	open (PG(2,12))	?
14	2	6	6	equivalent to Paley graph $\{6,3;1,3\}$	1
14	3	4	4	Mathon's construction	≥ 1
14	6	2	2	Mathon's construction	≥ 1
16	2	6	8	[dCMM], [So] and [Th1]	1
16	2	8	6	unique two-graph, i.e., $\frac{1}{2}H(6,2)$	1
16	4	2	4	GQ(3,5)$\backslash$ spread	≥ 5
16	6	4	2	GQ(5,3)$\backslash$ spread	≥ 1
16	7	2	2	OPEN	?
16	8	0	2		≥ 1
17	3	5	5	Mathon's construction	≥ 1
17	5	3	3	GQ(4,4)$\backslash$ unique spread	≥ 2
17	15	1	1	equivalent to PG(2,16), Mathon's construction	≥ 1
18	2	8	8	Mathon's construction	1
18	4	4	4	Mathon's construction	≥ 1
18	8	2	2	Mathon's construction	≥ 1
19	4	2	5	[Hae2] (GQ(3,6) does not exist	0
19	7	5	2	[Go4] (GQ(6,3) does not exist	0
19	17	1	1	open (PG(2,18))	?

Antipodal covers of diameter 4

Let Γ be an antipodal distance-regular graph of diameter 4, with v vertices, and let r be the size of its antipodal classes.

The intersection array $\{b_0, b_1, b_2, b_3; c_1, c_2, c_3, c_4\}$ is determined by (k, a_1, c_2, r), and has the following form

$$\{k, k - a_1 - 1, (r - 1)c_2, 1; 1, c_2, k - a_1 - 1, k\},$$

A systematic approach:

– a list of all small feasible parameters

– Krein conditions and absolute bounds

Let $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$ be $\mathrm{ev}(\Gamma)$.

The antipodal quotient is $\mathrm{SRG}(v/r, k, a_1, rc_2)$,
$\theta_0 = k$, θ_2, θ_4 are the roots of

$$x^2 - (a_1 - rc_2)x - (k - rc_2) = 0$$

and θ_1, θ_3 are the roots of $\quad x^2 - a_1 x - k = 0.$

The following relations hold for the eigenvalues:

$$\theta_0 = -\theta_1\theta_3, \text{ and } (\theta_2 + 1)(\theta_4 + 1) = (\theta_1 + 1)(\theta_3 + 1).$$

The multiplicities are $m_0 = 1$, $m_4 = (v/r) - m_2 - 1$,

$$m_2 = \frac{(\theta_4 + 1)k(k - \theta_4)}{rc_2(\theta_4 - \theta_2)} \text{ and } m_{1,3} = \frac{(r-1)v}{r(2 + a_1\theta_{1,3}/k)}.$$

Parameters of the antipodal quotient can be expressed in terms of eigenvalues and r: $\quad k = \theta_0$,

$$a_1 = \theta_1 + \theta_3,\ b_1 = -(\theta_2 + 1)(\theta_4 + 1),\ c_2 = \frac{\theta_0 + \theta_2\theta_4}{r}.$$

The eigenvalues θ_2, θ_4 are integral, $\theta_4 \leq -2,\ 0 \leq \theta_2$, with $\theta_2 = 0$ iff Γ is bipartite.

Furthermore, $\theta_3 < -1$, and the eigenvalues θ_1, θ_3 are integral when $a_1 \neq 0$.

We define for $s \in \{0,1,2,3,4\}$ the symmetric 4×4 matrix $P(s)$ with its ij-entry being equal to $p_{ij}(s)$.

For $b_1 = k-1-\lambda$, $k_2 = rkb_1/\mu$,
$a_2 = k-\mu$ and $b_2 = (r-1)\mu/r$ we have

$$P(0) = \begin{pmatrix} k & 0 & 0 & 0 \\ & k_2 & 0 & 0 \\ & & (r-1)k & 0 \\ & & & r-1 \end{pmatrix},$$

$$P(1) = \begin{pmatrix} \lambda & b_1 & 0 & 0 \\ & k_2 - b_1 r & b_1(r-1) & 0 \\ & & \lambda(r-1) & r-1 \\ & & & 0 \end{pmatrix},$$

$$P(2) = \begin{pmatrix} \mu/r & a_2 & b_2 & 0 \\ & k_2 - r(a_2+1) & (r-1)(k-\mu) & r-1 \\ & & b_2(r-1) & 0 \\ & & & 0 \end{pmatrix},$$

$$P(3) = \begin{pmatrix} 0 & b_1 & \lambda & 1 \\ & k_2 - rb_1 & b_1(r-1) & 0 \\ & & \lambda(r-2) & r-2 \\ & & & 0 \end{pmatrix},$$

$$P(4) = \begin{pmatrix} 0 & 0 & k & 0 \\ & k_2 & 0 & 0 \\ & & k(r-2) & 0 \\ & & & r-2 \end{pmatrix}.$$

The matrix of eigenvalues $P(\Gamma)$ (with $\omega_j(\theta_i)$ being its ji-entry) has the following form:

$$P(\Gamma) = \begin{pmatrix} 1 & \theta_0 & \theta_0 b_1/c_2 & \theta_0(r-1) & r-1 \\ 1 & \theta_1 & 0 & -\theta_1 & -1 \\ 1 & \theta_2 & -r(\theta_2+1) & \theta_2(r-1) & r-1 \\ 1 & \theta_3 & 0 & -\theta_3 & -1 \\ 1 & \theta_4 & -r(\theta_4+1) & \theta_4(r-1) & r-1 \end{pmatrix}$$

Theorem. (JK 1995).

Γ *antipodal distance-regular graph, diam* 4, *and eigenvalues* $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$.

Then $q^2_{11},\ q^3_{12},\ q^4_{13},\ q^2_{22},\ q^4_{22},\ q^3_{23},\ q^4_{24},\ q^4_{33} > 0$,
$r = 2$ *iff* $q^1_{11} = 0$ *iff* $q^3_{11} = 0$ *iff* $q^3_{13} = 0$ *iff* $q^3_{33} = 0$,
$q^2_{12} = q^4_{12} = q^4_{14} = q^3_{22} = q^4_{23} = q^4_{34} = 0$ *and*

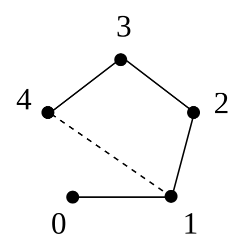
(i) $(\theta_4 + 1)^2(k^2 + \theta_2^3) \ \geq\ (\theta_2 + 1)(k + \theta_2\theta_4)$,
with equality iff $q^2_{22} = 0$,

(ii) $(\theta_2 + 1)^2(k^2 + \theta_4^3) \ \geq\ (\theta_4 + 1)(k + \theta_2\theta_4)$,
with equality iff $q^4_{44} = 0$,

(iii) $\theta_3^2 \ \geq\ -\theta_4$, *with equality iff* $q^4_{11} = 0$.

Let E be a primitive idempotent of a distance-regular graph of diameter d. The **representation diagram** $\boldsymbol{\Delta_E}$ is the undirected graph with vertices $0, 1, \ldots d$, where we join two distinct vertices i and j whenever $q_{ij}^s = q_{ji}^s \neq 0$.

Recall Terwilliger's characterization of Q-polynomial association schemes that a d-class association scheme is Q-polynomial iff the representation diagram a minimal idempotent, is a path. For $s = 1$ and $r = 2$ we get the following graph:



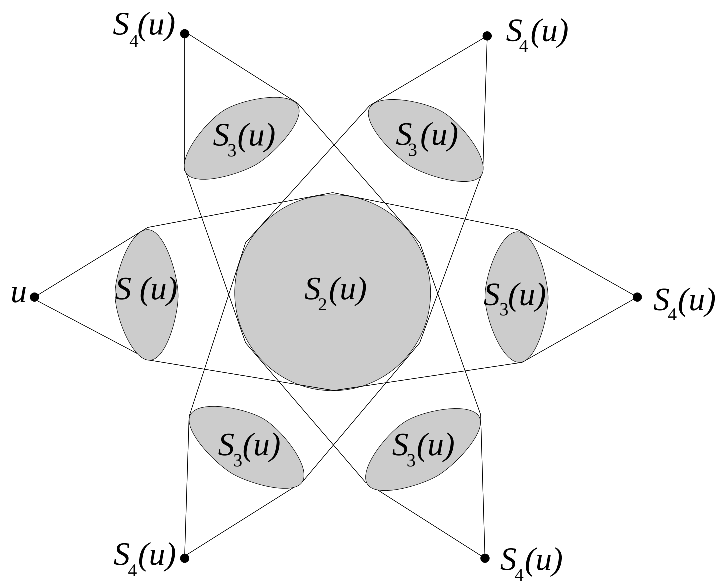
Based on the above information we have:

Corollary. Γ *antipodal, distance-regular graph with diam.* 4. *TFAE*

(i) Γ *is Q-polynomial.*

(ii) $r = 2$ *and* $q_{11}^4 = 0$.

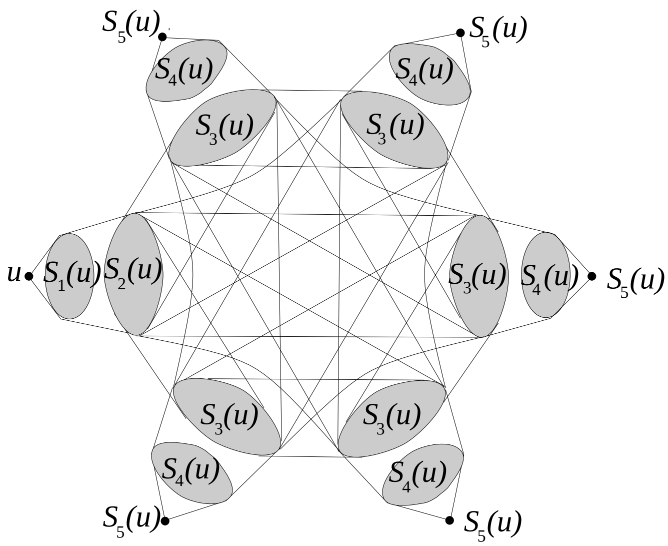
Suppose (i)-(ii) hold, then $\theta_0, \theta_1, \theta_2, \theta_3, \theta_4$ *is a unique Q-polynomial ordering, and* $q_{ij}^h = 0$ *when* $i + j + h$ *is odd, i.e., the Q-polynomial structure is dual bipartite.*



An antipodal distance-regular graph of diameter 4
(the distance partition corresponding to an antipodal class).

#	Γ	n	k	λ	μ	H	r	$r.n$
1	! Folded 5-cube	16	5	0	2	! Wells graph	2	32
2	! $\overline{T(6)}$	15	6	1	3	! 3.Sym(6).2	3	45
3	! $\overline{T(7)}$	21	10	3	6	! 3.Sym(7)	3	63
4	folded J(8,4)	35	16	6	8	! Johnson graph J(8,4)	2	70
5	! truncated 3-Golay code	81	20	1	6	shortened 3-Golay code	3	243
6	! folded halved 8-cube	64	28	12	12	! halved 8-cube	2	128
7	$S_2(S_2(McL.))$	105	32	4	12	S_2(Soicher1 graph)	3	315
8	Zara graph (126,6,2)	126	45	12	18	$3.O_6^-(3)$	3	378
9	! S_2(McLaughlin graph) [Br3]	162	56	10	24	! Soicher1 graph	3	486
10	hyperbolic pts. of PG(6,3)	378	117	36	36	$3.O_7(3)$	3	1134
11	Suzuki graph	1781	416	100	96	Soicher2 [Soi]	3	5346
12		306936	31671	3510	3240	$3.Fi_{24}^-$	3	

Non-bipartite antipodal distance-regular graphs of diameter 4.



#	Γ	n	k	λ	μ	H	r	t	$r.n$
1	! Petersen graph	10	3	0	1	! Dodecahedron	2	1	20
2	3-Golay code	243	22	1	2	short. ext. 3-Golay code	3	9	729
3	folded Johnson graph $J(10,5)$	126	25	8	8	! Johnson graph $J(10,5)$	2	9	252
4	folded halved 10-cube	256	45	16	6	! halved 10-cube	2	15	512

Non-bipartite antipodal distance-regular graphs of diameter 5.