**Theorem (Gardiner, 1974).** *If $H$ is antipodal $r$-cover of $G$, then $\iota(H)$ is (almost) determined by $\iota(G)$ and $r$,*

$$D_H \in \{2d_\Gamma, 2d_\Gamma + 1\} \quad and \quad 2 \leq r \leq k,$$

*and*

$$b_i = c_{D-i} \ for \ \ i = 0, \ldots, D, \ i \neq d, \qquad r = 1 + \frac{b_d}{c_{D-d}}.$$

**Lemma.** *A distance-regular antipodal graph $\Gamma$ of diameter $d$ is a cover of its antipodal quotient with components of $\Gamma_d$ as its fibres unless $d = 2$.*

**Lemma.** $\Gamma$ *antipodal distance-regular, diameter* $d$. *Then a vertex* $x$ *of* $\Gamma$, *which is at distance* $i \leq \lfloor d/2 \rfloor$ *from one vertex in an antipodal class, is at distance* $d-i$ *from all other vertices in this antipodal class. Hence*

$$\Gamma_{d-i}(x) = \cup\{\Gamma_d(y) \mid y \in \Gamma_i(x)\} \qquad \text{for } 0 \leq i \leq \lfloor d/2 \rfloor\,.$$

For each vertex $u$ of a cover $H$ we denote the fibre which contains $u$ by $F(u)$.

A **geodesic** in a graph $G$ is a path $g_0, \ldots, g_t$, where $\mathrm{dist}(g_0, g_t) = t$.

**Theorem.** *$G$ distance-regular, diameter $d$ and parameters $b_i$, $c_i$; $H$ its $r$-cover of diameter $D > 2$. Then the following statements are equivalent:*

**(i)** *The graph $H$ is antipodal with its fibres as the antipodal classes (hence an antipodal cover of $G$) and each geodesic of length at least $\lfloor (D+1)/2 \rfloor$ in $H$ can be extended to a geodesic of length $D$.*

**(ii)** *For any $u \in V(H)$ and $0 \leq i \leq \lfloor D/2 \rfloor\}$ we have*

$$S_{D-i}(u) = \cup\{F(v)\backslash\{v\} \ : \ v \in S_i(u)\}.$$

**(iii)** *The graph $H$ is distance-regular with $D \in \{2d, 2d+1\}$ and intersection array*

$$\Big\{b_0, \ldots, b_{d-1}, \frac{(r-1)c_d}{r}, c_{d-1}, \ldots, c_1;$$

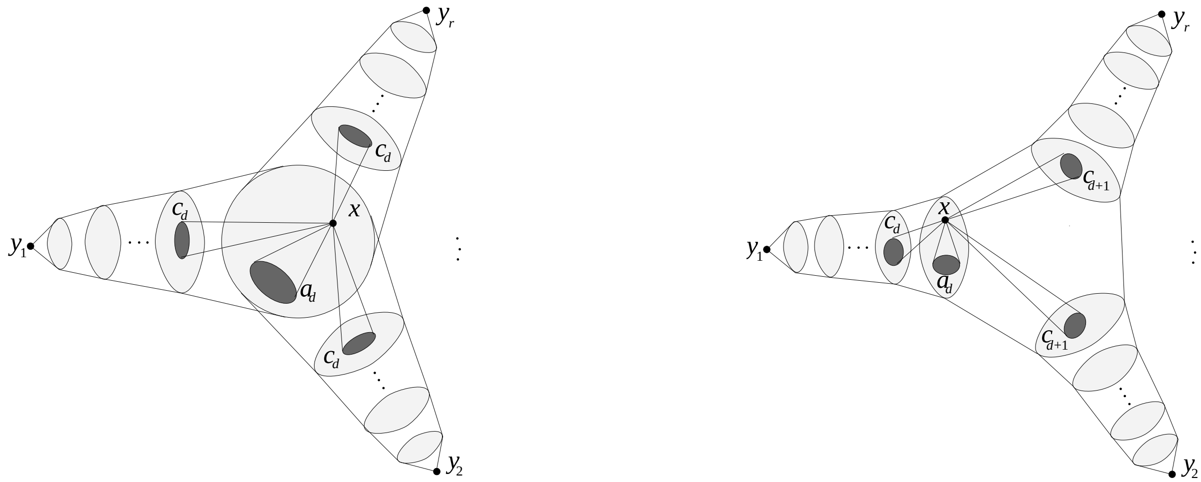
$$c_1, \ldots, c_{d-1}, \frac{c_d}{r}, b_{d-1}, \ldots, b_0\Big\} \quad \textit{for } D \textit{ even,}$$

*and*

$$\Big\{b_0, \ldots, b_{d-1}, (r-1)t, c_d, \ldots, c_1;$$

$$c_1, \ldots, c_d, t, b_{d-1}, \ldots, b_0\Big\}$$

*for $D$ odd and some integer $t$.*




The distance distribution corresponding to the antipodal class $\{y_1, \ldots, y_r\}$ in the case when $d$ is even (left) and the case when $d$ is odd (right).

Inside this partition there is a partition of the neighbourhood of the vertex $x$.

For $\theta \in \text{ev}(\Gamma)$ and associated primitive idempotent $E$:

$$E = \frac{m_\theta}{|V\Gamma|} \sum_{h=0}^{d} \omega_h A_h \quad (0 \le i \le d),$$

$\omega_0, \ldots, \omega_d$ is the **cosine sequence** of $E$ (or $\theta$).

**Lemma.** *$\Gamma$ distance-regular, diam. $d \ge 2$, $E$ is a primitive idempotent of $\Gamma$ corresponding to $\theta$, $\omega_0, \ldots, \omega_d$ is the cosine sequence of $\theta$.*
*For $x, y \in V\Gamma$, $i = \partial(x, y)$ we have*
*(i)* $\langle Ex, Ey \rangle =$ *$xy$-entry of* $E = \omega_i \dfrac{m_\theta}{|V\Gamma|}$.
*(ii)* $\omega_0 = 1$ *and* $c_i\omega_{i-1} + a_i\omega_i + b_i\omega_{i+1} = \theta\omega_i$ *for* $0 \le i \le d$.

$$\omega_1 = \frac{\theta}{k}, \qquad \omega_2 = \frac{\theta^2 - a_1\theta - k}{kb_1}$$

Using the Sturm's theorem for the sequence

$$w_i(x) = b_0 \dots b_i \, \omega_i(x)$$

we obtain

**Theorem.** *Let $\theta_0 \geq \cdots \geq \theta_d$ be the eigenvalues of a distance regular graph. The sequence of cosines corresponding to the $i$-th eigenvalue $\theta_i$ has precisely $i$ sign changes.*

**Theorem.** *Let $\Gamma$ be a distance regular graph and $H$ a distance regular antipodal r-cover of $G$. Then every eigenvalue $\theta$ of $\Gamma$ is also an eigenvalue of $H$ with the same multiplicity.*

PROOF. Let $H$ has diameter $D$, and $\Gamma$ $n$ vertices, so $H_D = n \cdot K_r$ ($K_r$'s are corrsp. to the fibres of $H$).

Therefore, $H_D$ has for eigenvalues $r - 1$
with multiplicity $n$ and $-1$ with multiplicity $nr - n$.

The eigenvectors corresponding to eigenvalue $r - 1$ are constant on fibres and those corresponding to $-1$ sum to zero on fibres.

Take $\theta$ to be an eigenvalue of $H$, which is also an eigenvalue of $\Gamma$.

An eigenvector of $\Gamma$ corresponding to $\theta$ can be extended to an eigenvector of $H$ which is constant on fibres.

We know that the eigenvectors of $H$ are also the eigenvectors of $H_D$, therefore, we have $v_D(\theta) = r - 1$.

So we conclude that all the eigenvectors of $H$ corresponding to $\theta$ are constant on fibres and therefore give rise to eigenvectors of $\Gamma$ corresponding to $\theta$. ■

## Connections

– **projective and affine planes**, for $D = 3$, or $D = 4$ and $r = k$ (covers of $K_n$ or $K_{n,n}$),

– **Two graphs** ($Q$-polynomial), for $D = 3$ and $r = 2$,

– **Moore graphs**, for $D = 3$ and $r = k$,

– **Hadamard matrices**, $D = 4$ and $r = 2$ (covers of $K_{n,n}$),

– **group divisible resolvable designs**, $D = 4$ (cover of $K_{n,n}$),

– coding theory (perfect codes),
– group theory (class. of finite simple groups),
– orthogonal polynomials.

# Tools:

– graph theory, counting,
– matrix theory (rank mod $p$),
– eigenvalue techniques,
– representation theory of graphs,
– geometry (Euclidean and finite),
– algebra and association schemes,
– topology (covers and universal objects).

## Goals:

– structure of antipodal covers,
– new infinite families,
– nonexistence and uniqueness,
– characterization,
– new techniques
(which can be applied to drg or even more general)

Difficult problems:

Find a 7-cover of $K_{15}$.
Find a double-cover of Higman-Sims graph ($\{22, 21; 1, 6\}$).