VI. Equitable partitions

- definition
- quotients
- eigenvectors
- (antipodal) covers

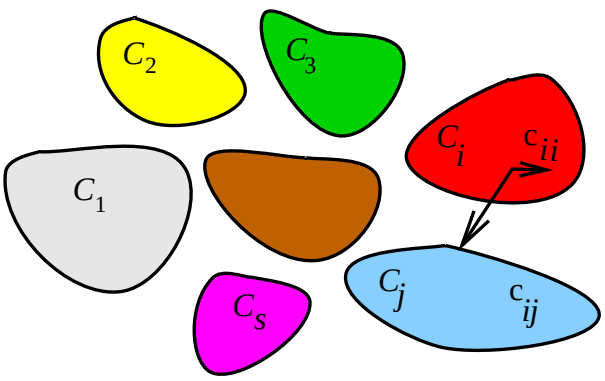
$$\begin{pmatrix} 1 & a & b & c & d & e \\ a & 1 & a & b & c & d \\ b & a & 1 & a & b & c \\ c & b & a & 1 & a & b \\ d & c & b & a & 1 & a \\ e & d & c & b & a & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1+a & a+b & b+c \\ a+b & 1+c & a+d \\ b+c & a+d & 1+e \end{pmatrix} \begin{pmatrix} 1-a & a-b & b-c \\ a-b & 1-c & a-d \\ b-c & a-d & 1-e \end{pmatrix}$$

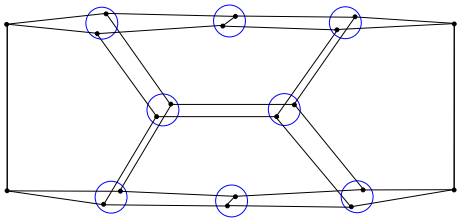
An **equitable partition** of a graph Γ is a partition of the vertex set $V(\Gamma)$ into **parts** $C_1, C_2, \ldots, C_s$ s.t.

(a) vertices of each part C_i induce a *regular* graph,

(b) edges between C_i and C_j induce a *half-regular* graph.



Example: the dodecahedron

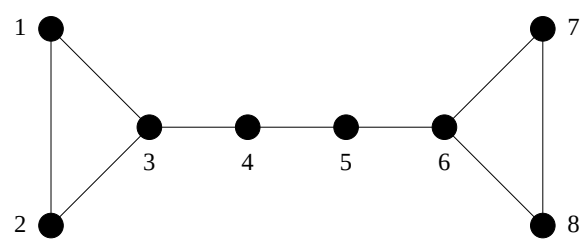


Numbers c_{ij} are the parameters of the partition.

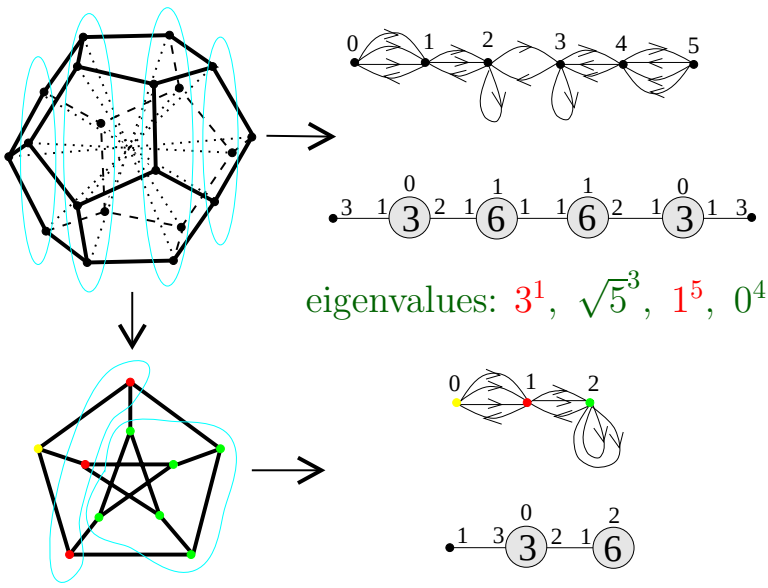
Orbits of a group acting on Γ form an equitable partition.

But not all equitable partitions come from groups:

$$\{\{1, 2, 4, 5, 7, 8\}, \{3, 6\}\}.$$



Equitable partitions give rise to **quotient** graphs $\boldsymbol{G/\pi}$, which are directed multigraphs with cells as vertices and c_{ij} arcs going from C_i to C_j.



eigenvalues: $3^1,\ \sqrt{5}^3,\ 1^5,\ 0^4,\ -2^4,\ -\sqrt{5}^3$

Set $\boldsymbol{X} := V\Gamma$ and $\boldsymbol{n} := |X|$. Let $\boldsymbol{V} = \mathbb{R}^n$ be the vector space over $\mathbb{R}$ consisting of all column vectors whose coordinates are indexed by X.

For a subset $S \subseteq X$ let its **characteristic vector** be an element of V, whose coordinates are equal 1 if they correspond to the elements of S and 0 otherwise.

Let $\pi = \{C_1, \ldots, C_s\}$ be a partition of X.
The **characteristic matrix** P of π is $(n \times s)$ matrix, whose column vectors are the characteristic vectors of the parts of π (i.e., $P_{ij} = 1$ if $i \in C_j$ and 0 otherwise).

Let $\mathbf{Mat_X(\mathbb{R})}$ be the $\mathbb{R}$-algebra consisting of all real matrices, whose rows and columns are indexed by X. Let $\boldsymbol{A} \in \mathrm{Mat}_X(\mathbb{R})$ be the adjacency matrix of Γ.

$\mathrm{Mat}_X(\mathbb{R})$ acts on V by left multiplication.

Theorem. *Let π be a partition of $V\Gamma$ with the characteristic matrix P. TFAE*

(i) π equitable,

(ii) $\exists$ a $s \times s$ matrix B s.t. $A(\Gamma)P = PB$

(iii) the span(col(P)) is $A(\Gamma)$-invariant.

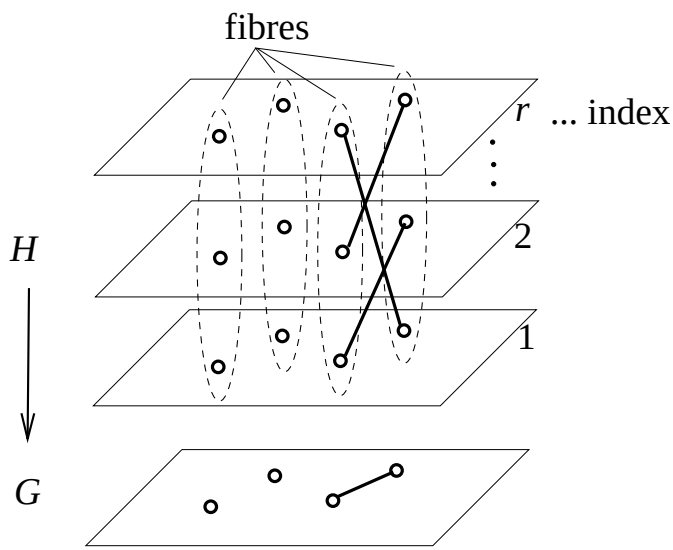
If π is equitable then $B = A(\Gamma/\pi)$.

Theorem. *Assume $AP = PB$.*
(a) If $B\boldsymbol{x} = \theta\boldsymbol{x}$, then $AP\boldsymbol{x} = \theta P\boldsymbol{x}$.
(b) If $A\boldsymbol{y} = \theta\boldsymbol{y}$, then $\boldsymbol{y}^T PB = \theta\boldsymbol{y}^T P$.
(c) The characteristic polynomial of matrix B divides the characteristic polynomial of matrix A.

An eigenvector x of Γ/π corresponding to θ extends to an eigenvector of Γ, which is constant on parts, so

$$m_\theta(\Gamma/\pi) \leq m_\theta(\Gamma).$$

$\tau \in \mathrm{ev}(\Gamma)\backslash\mathrm{ev}(\Gamma/\pi) \implies$ each eigenvector of Γ corresponding to τ sums to zero on each part.

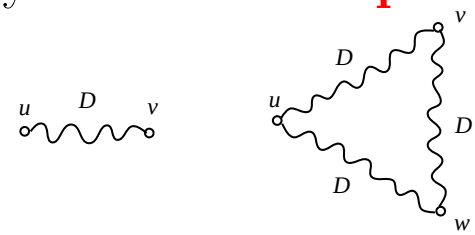


H is an ***r*-cover** if there is a partition of $V(H)$ into independent sets, called **fibres**, such that there is either a matching or nothing between any two fibres.

$\Gamma = H/\pi \ldots$ **quotient** (corresponding to π)

H graph, D diameter

If being at distance 0 or D is an equivalence relation on $V(H)$, we say that H is **antipodal**.



If an antipodal graph H covers H/π and π consists of antipodal classes, then H is called **antipodal cover**.