The **eigenmatrices** of the associative scheme $\mathcal{A}$ are $(d+1)$-dimensional square matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ defined by

$$(P)_{ij} = p_i(j) \quad \text{and} \quad (Q)_{ij} = q_i(j).$$

The eigenvalue $p_i(1)$ of the matrix A_1 has multiplicity $m_i = q_i(0)$ and is equal to the rank(E_i).

By Theorem (b) and Corollary (b), $\quad PQ = nI$.

It is not difficult to verify also

$$\Delta_k Q = (\Delta_m P)^T,$$

where Δ_k and Δ_m are the diagonal matrices with entries $(\Delta_k)_{ii} = k_i$ and $(\Delta_m)_{ii} = m_i$.

Using the eigenvalues we can express all intersection numbers and Krein parameters.

For example, if we multiply the equality in Corollary (a) by E_h, we obtain

$$q_{ij}^h \, E_h = nE_h(E_i \circ E_j),$$

i.e.,

$$\begin{aligned} q_{ij}^h \; &= \; \frac{n}{m_h} \, \mathrm{trace}(E_h(E_i \circ E_j)) && (7) \\ &= \; \frac{n}{m_h} \, \mathrm{sum}(E_h \circ E_i \circ E_j), && (8) \end{aligned}$$

where the sum of a matrix is equal to the sum of all of its elements.

By Corollary (b), it follows also

$$E_i \circ E_j \circ E_h = \frac{1}{n^3} \sum_{\ell=0}^{d} q_i(\ell)\, q_j(\ell)\, q_h(\ell)\, A_\ell,$$

therefore, by $\Delta_k Q = (\Delta_m P)^T$, we obtain

$$\begin{aligned} q_{ij}^h &= \frac{1}{nm_h} \sum_{\ell=0}^{d} q_i(\ell)\, q_j(\ell)\, q_h(\ell)\, k_\ell \\ &= \frac{m_i m_j}{n} \sum_{\ell=0}^{d} \frac{p_\ell(i)\, p_\ell(j)\, p_\ell(h)}{k_{\ell^2}.} \end{aligned}$$

Krein parameters satisfy the so-called **Krein conditions**:

Theorem [Scott].
Let $\mathcal{A}$ be an associative scheme with n vertices and $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ the standard basis in $\mathbb{R}^n$. Then

$$\boxed{q_{ij}^h \geq 0.}$$

Moreover, for $\boldsymbol{v} = \sum_{i=1}^{n} \boldsymbol{e}_i \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_i$, we have

$$q_{ij}^h = \frac{n}{m_h} \left\| (E_i \otimes E_j \otimes E_h)\boldsymbol{v} \right\|^2,$$

and $q_{ij}^h = 0$ iff $(E_i \otimes E_j \otimes E_h)\boldsymbol{v} = 0$.

PROOF (Godsil's sketch). Since the matrices E_i are pairwise orthogonal idempotents, we derive from Corollary (b) (by multiplying by E_h)

$$(E_i \circ E_j)E_h \;\; = \;\; \frac{1}{n}\, q_{ij}^h E_h.$$

Thus q_{ij}^h/n is an eigenvalue of the matrix $E_i \circ E_j$ on a subspace of vectors that are determined by the columns of E_h.

The matrices E_i are positive semidefinite (since they are symmetric, and all their eigenvalues are 0 or 1).

On the other hand, the Schur product of semidefinite matrices is again semidefinite, so the matrix $E_i \circ E_j$ has nonnegative eigenvalues. Hence, $q_{ij}^h \geq 0$.

By (7) and the well known tensor product identity

$$(A \otimes B)(x \otimes y) = Ax \otimes By$$

for $A, B \in \mathbb{R}^{n \times n}$ and $x, y \in \mathbb{R}^n$, we obtain

$$\begin{aligned} q_{ij}^h &= \frac{n}{m_h} \operatorname{sum}(E_i \circ E_j \circ E_h) \\ &= \frac{n}{m_h} \boldsymbol{v}^T (E_i \otimes E_j \otimes E_h) \boldsymbol{v}. \end{aligned}$$

Now the statement follows from the fact that

$$E_i \otimes E_j \otimes E_h$$

is a symmetric idempotent. ■

Another strong criterion for an existence of associative schemes is an **absolute bound**, that bounds the rank of the matrix $E_i \circ E_j$.

Theorem. *Let $\mathcal{A}$ be a d-class associative scheme. Then its multiplicities m_i, $1 \le i \le d$, satisfy inequalities*

$$\sum_{q_{ij}^h \neq 0} m_h \le \begin{cases} m_i m_j & \text{if } i \neq j, \\ \frac{1}{2} m_i (m_i + 1) & \text{if } i = j. \end{cases}$$

Proof (sketch). The LHS is equal to the rank$(E_i \circ E_j)$ and is greater or equal to the

$$\mathrm{rank}(E_i \otimes E_j) = m_i m_j.$$

Suppose now $i = j$. Among the rows of the matrix E_i we can choose m_i rows that generate all the rows.

Then the rows of the matrix $E_i \circ E_i$, whose elements are the squares of the elements of the matrix E_i, are generated by

$$m_i + \binom{m_i}{2} \text{ rows,}$$

that are the Schur products of all the pairs of rows among all the m_i rows. ■

An association scheme $\mathcal{A}$ is ***P*-polynomial** (called also **metric**) when there there exists a permutation of indices of A_i's, s.t.

$$\exists \text{ polynomials } p_i \text{ of degree } i \text{ s.t. } A_i = p_i(A_1),$$

i.e., the intersection numbers satisfy the **Δ-condition**

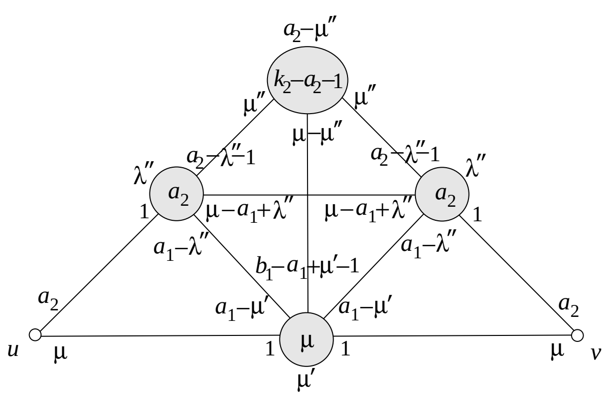
(that is, $\forall\ i, j, h \in \{0, \ldots, d\}$

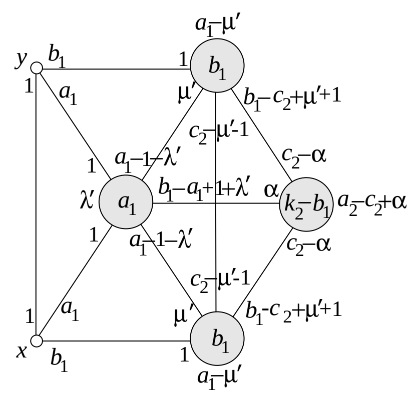
- $p_{ij}^h \neq 0$ implies $h \leq i + j$ and
- $p_{ij}^{i+j} \neq 0$).

An associative scheme $\mathcal{A}$ is ***Q*-polynomial** (called also **cometric**) when there exists a permutation of indices of E_i's, s.t. the Krein parameters q_{ij}^h satisfy the Δ-condition.

Theorem. **[Cameron, Goethals and Seidel]**

In a strongly regular graph vanishing of either of Krein parameters q_{11}^1 and q_{22}^2 implies that first and second subconstituent graphs are strongly regular.





SRG$(162, 56, 10, 24)$, denoted by Γ,
is unique by Cameron, Goethals and Seidel.

vertices: special vertex ∞,
56 hyperovals in PG$(2, 4)$ in a $L_3(4)$-orbit,
105 flags of PG$(2, 4)$

adjacency: ∞ is adjacent to the hyperovals

$$\begin{aligned} \text{hyperovals } \mathcal{O} \sim \mathcal{O}' &\iff \mathcal{O} \cap \mathcal{O}' = \emptyset \\ (p, L) \sim \mathcal{O} &\iff |\mathcal{O} \cap L \backslash \{p\}| = 2 \\ (p, L) \sim (q, M) &\iff p \neq q,\ L \neq M \text{ and} \\ & \qquad (p \in M \text{ or } q \in L). \end{aligned}$$

The hyperovals induce the Gewirtz graph,
i.e., the unique SRG(56,10,0,2))
and the flags induce a SRG(105,32,4,12).

Theorem. *Assume $AP = PB$.*
(a) If $B\boldsymbol{x} = \theta\boldsymbol{x}$, then $AP\boldsymbol{x} = \theta P\boldsymbol{x}$.
(b) If $A\boldsymbol{y} = \theta\boldsymbol{y}$, then $\boldsymbol{y}^T PB = \theta\boldsymbol{y}^T P$.
(c) The characteristic polynomial of matrix B divides the characteristic polynomial of matrix A.

An eigenvector x of Γ/π corresponding to θ extends to an eigenvector of Γ, which is constant on parts, so

$$m_\theta(\Gamma/\pi) \le m_\theta(\Gamma).$$

$\tau \in \mathrm{ev}(\Gamma)\backslash\mathrm{ev}(\Gamma/\pi) \implies$ each eigenvector of Γ corresponding to τ sums to zero on each part.