Hamming scheme $H(d, n)$

Let $d, n \in \mathbb{N}$ and $\Sigma = \{0, 1, \ldots, n-1\}$.

The vertex set of the association scheme $H(d, n)$ are all d-tuples of elements on Σ. Assume $0 \leq i \leq d$.

Vertices x and y are in i-th relation iff they differ in i places.

We obtain a d-class association scheme on n^d vertices.

Bilinear Forms Scheme $\mathcal{M}_{d\times m}(q)$

(a variation from linear algebra) Let $d, m \in \mathbb{N}$ and q a power of some prime.

All $(d \times m)$-dim. matrices over GF(q) are the vertices of the scheme,

two being in i-th relation, $0 \le i \le d$,
when the rank of their difference is i.

Johnson Scheme $J(n, d)$

Let $d, n \in \mathbb{N}$, $d \le n$ and X a set with n elements.

The vertex set of the association scheme $J(n, d)$ are all d-subsets of X.

Vertices x and y are in i-th $0 \le i \le \min\{d, n-d\}$, relation iff their intersection has $d - i$ elements.

We obtain an association scheme with $\min\{d, n-d\}$ classes and on $\binom{n}{d}$ vertices.

q-analog of Johnson scheme $J_q(n, d)$ (Grassman scheme)

The vertex set consists of all d-dim. subspaces of n-dim. vector space V over GF(q).

Two subspaces A and B of dim. d are in i-th relation, $0 \leq i \leq d$, when $\dim(A \cap B) = d - i$.

Cyclomatic scheme

Let q be a prime power and d a divisor of $q-1$.

Let C_1 be a subgroup of the multiplicative group of the finite field $\mathrm{GF}(q)$ with index d, and let $C_1, \dots, C_d$ be the cosets of the subgroup C_1.

The vertex set of the scheme are all elements of $\mathrm{GF}(q)$, x and y being in i-th relation when $x - y \in C_i$ (and in 0 relation when $x = y$).

We need $-1 \in C_1$ in order to have symmetric relations, so $2 \,|\, d$, if q is odd.

How can we verify if some set of matrices represents an association scheme?

The condition (b) does not need to be verified directly.

It suffices to check that the RHS of (4) is independent of the vertices (without computing p_{ij}^h).

We can use *symmetry*.

Let X be the vertex set and $\Gamma_1, \ldots, \Gamma_d$ the set of graphs with $V(\Gamma_i) = X$ and whose adjacency matrices together with the identity matrix satisfies the condition (a).

Symmetry

An **automorphism** of this set of graphs is a permutation of vertices, that preserves adjacency.

Adjacency matrices of the graphs $\Gamma_1, \ldots, \Gamma_d$, together with the identity matrix is an association scheme if

> $\forall i$ the automorphism group acts transitively on pairs of vertices that are adjacent in Γ_i

(this is a sufficient condition).

Primitivity and imprimitivity

A d-class association scheme is **primitive**, if all its graphs Γ_i, $1 \leq i \leq d$, are connected, and **imprimitive** othervise.

The trivial scheme is primitive.

Johnson scheme $J(n,d)$ is primitive iff $n \neq 2d$.
In the case $n = 2d$ the graph Γ_d is disconnected.

Hamming scheme $H(d,n)$ is primitive iff $n \neq 2$.
In the case $n = 2$ the graphs Γ_i, $1 \leq i \leq \lfloor d/2 \rfloor$, and the graph Γ_d are disconnected.

Let $\{A_0, \ldots, A_d\}$ be a d-class associative scheme $\mathcal{A}$ and let π be a partition of $\{1, \ldots, d\}$ on $m \in \mathbb{N}$ nonempty cells. Let $A'_1, \ldots, A'_m$ be the matrices of the form

$$\sum_{i \in C} A_i,$$

where C runs over all cells of partition π, and set $A'_0 = I$. These binary matrices are the elements of the Bose-Mesner algebra $\mathcal{M}$, they commute, and their sum is J.

Very often the form an associative scheme, denoted by $\mathcal{A}'$, in which case we say that $\mathcal{A}'$ was obtained from $\mathcal{A}$ by **merging** of classes (also by **fusion**).

For $m = 1$ we obtain the trivial associative scheme.

Brouwer and Van Lint used merging to construct some new 2-class associative schemes (i.e., $m = 2$).

For example, in the Johnson scheme $J(7, 3)$ we merge A_1 and A_3 to obtain a strongly regular graph, which is the line graph of $PG(3, 2)$.

Two bases and duality

Theorem. *Let $\mathcal{A} = \{A_0, \dots, A_d\}$ be an associative scheme on n vertices. Then there exists orthogonal idempotent matrices $E_0, \dots, E_d$ and $p_i(j)$, such that*

(a) $$\sum_{j=0}^{d} E_j = I,$$

(b) $$A_i E_j = p_i(j) E_j,$$

(c) $$E_0 = \frac{1}{n} J,$$

(d) matrices $E_0, \dots, E_d$ are a basis of a $(d+1)$-dim. vector space, generated by $A_0, \dots, A_d$.

PROOF. Let $i \in \{1, 2, \ldots, d\}$. From the spectral analysis of normal matrices we know that $\forall A_i$ there exist pairwise orthogonal idempotent matrices Y_{ij} and real numbers θ_{ij}, such that $A_i Y_{ij} = \theta_{ij} Y_{ij}$ and

$$\sum_j Y_{ij} = I. \tag{5}$$

Furthermore, each matrix Y_{ij} can be expressed as a polynomial of the matrix A_i.

Since $\mathcal{M}$ is a commutative algebra, the matrices Y_{ij} commute and also commute with matrices $A_0, \ldots, A_d$.

Therefore, each product of this matrices is an idempotent matrix (that can be also 0).

We multiply equations (5) for $i = 1, \dots, d$. to obtain an equation of the following form

$$I = \sum_j E_j, \tag{6}$$

where each E_j is an idempotent that is equal to a product of d idempotents Y_{ik_i}, where Y_{ik_i} is an idempotent from the spectral decompozition of A_i.

Hence, the idempotents E_j are pairwise orthogonal, and for each matrix A_i there exists $p_i(j) \in \mathbb{R}$, such that $A_i E_j = p_i(j) E_j$.

Therefore,

$$A_i = A_i I = A_i \sum_j E_j = \sum_j p_i(j) E_j.$$

This tells us that each matrix A_i is a linear combination of the matrices E_j.

Since the nonzero matrices E_j are pairwise othogonal, they are also linearly independent.

Thus they form a basis of the BM-algebra $\mathcal{M}$, and there is exactly $d+1$ nonzero matrices among E_j's.

The proof of (c) is left for homework. $\blacksquare$

The matrices $E_0, \ldots, E_d$ are called **primitive idempotents** of the associative scheme $\mathcal{A}$. **Schur** (or Hadamard) product of matrices is an entry-wise product. denoted by "$\circ$". Since $A_i \circ A_j = \delta_{ij} A_i$, the BM-algebra is closed for Schur product.

The matrices A_i are pairwise othogonal idempotents for Schure multiplication, so they are also called **Schur idempotents** of $\mathcal{A}$.

Since the matrices $E_0, \ldots, E_d$ are a basis of the vector space spanned by $A_0, \ldots, A_d$, also the following statement follows.

Corollary. *Let $\mathcal{A} = \{A_0, \ldots, A_d\}$ be an associative scheme and $E_0, \ldots, E_d$ its primitive idempotents. Then $\exists\ q_{ij}^h \in \mathbb{R}$ and $q_i(h) \in \mathbb{R}$ $(i, j, h \in \{0, \ldots, d\})$, such that*

(a) $$E_i \circ E_j = \frac{1}{n} \sum_{h=0}^{d} q_{ij}^h E_h,$$

(b) $$E_i = \frac{1}{n} \sum_{h=0}^{d} q_i(h) A_h,$$

(c) *matrices A_i have at most $d+1$ distinct eigenvalues.* ■

There exists a basis of $d+1$ (orthogonal) primitive idempotents E_i of the BM-algebra $\mathcal{M}$ such that

$$E_0 = \frac{1}{n}J \quad \text{and} \quad \sum_{i=0}^{d} E_i = I,$$

$$E_i \circ E_j \;=\; \frac{1}{n}\sum_{h=0}^{d} q_{ij}^h E_h, \quad A_i \;=\; \sum_{h=0}^{d} p_i(h) E_h$$

$$\text{and} \quad E_i \;=\; \frac{1}{n}\sum_{h=0}^{d} q_i(h) A_h \;\; (0 \le i, j \le d),$$

The parameters q_{ij}^h are called **Krein parameters**, $p_i(0), \ldots, p_i(d)$ are **eigenvalues** of matrix A_i, and $q_i(0), \ldots, q_i(d)$ are the **dualne eigenvalues** of E_i.