Classical generalized quadrangles

due to J. Tits (all associated with classical groups)

An **orthogonal** generalized quadrangle $\boldsymbol{Q(d,q)}$ is determined by isotropic points and lines of a nondegenerate quadratic form in

$$\mathrm{PG}(d,q), \quad \text{for } d \in \{3,4,5\}.$$

For $d = 3$ we have $t = 1$. An orthogonal generalized quadrangle $Q(4, q)$ has parameters $\boldsymbol{(q, q)}$.

Its dual is called **symplectic** (or **null**)
generalized quadrangle $\boldsymbol{W(q)}$
(since it can be defined on points of $\mathrm{PG}(3, q)$, together with the self-polar lines of a null polarity),

and it is for even q isomorphic to $Q(4, q)$.

Let H be a nondegenerate hermitian variety (e.g., $V(x_0^{q+1} + \cdots + x_d^{q+1})$) in $\mathrm{PG}(d, q^2)$.

Then its points and lines form a generalized quadrangle called a **unitary** (or **Hermitean**) generalized quadrangle $\boldsymbol{\mathcal{U}(d, q^2)}$.

A unitary generalized quadrangle $\mathcal{U}(3, q^2)$ has parameters $\boldsymbol{(q^2, q)}$ and is isomorphic to a dual of orthogonal generalized quadrangle $Q(5, q)$.

Finally, we describe one more construction (Ahrens, Szekeres and independently M. Hall)

Let $\mathcal{O}$ be a hyperoval of $\mathrm{PG}(2,q)$, $q=2^h$, i.e., (i.e., a set of $q+2$ points meeting $\forall$ line in 0 or 2 points), and imbed $\mathrm{PG}(2,q)=H$ as a plane in $\mathrm{PG}(3,q)=P$.

Define a generalized quadrangle $\boldsymbol{T_2^*(\mathcal{O})}$ with parameters

$$\boldsymbol{(q-1,q+1)}$$

by taking for points just the points of $P\backslash H$, and for lines just the lines of P which are not contained in H and meet $\mathcal{O}$ (necessarily in a unique point).

For a systematic combinatorial treatment of generalized quadrangles we recommend the book by **Payne and Thas**.

The order of each known generalized quadrangle or its dual is one of the following: $(s, 1)$ for $s \geq 1$;

$$(q, q),$$
$$(q, q^2),$$
$$(q^2, q^3),$$
$$(q - 1, q + 1),$$

where q is a prime power.

Small examples

$s = 2$: !(2,2), !(2,4)

$s = 3$: $(3,3) = W(3)$ or $Q(4,3)$,
$(3,5) = T_2^*(\mathcal{O})$,
$(3,9) = Q(5,3)$

$s = 4$: $(4,4) = W(4)$,

one known example for each (4,6), (4,8), (4,16)

existence open for (4,11), (4,12).

The flag geometry of a generalized polygon $\mathcal{G}$ has as pts the vertices of $\mathcal{G}$ (of both types), and as lines the flags of $\mathcal{G}$, with the obvious incidence.

It is easily checked to be a generalised $2n$-gon in which every line has two points; and any generalised $2n$-gon with two points per line is the flag geometry of a generalised n-gon.

Theorem (Feit and Higman). *A thick generalised n-gon can $\exists$ only for $n = 2, 3, 4, 6$ or 8.*

Additional information:

- *if $n = 4$ or $n = 8$, then $t \leq s^2$ and $s \leq t^2$;*
- *if $n = 6$, then st is a square and $t \leq s^3$, $s \leq t^3$.*
- *if $n = 8$, then $2st$ is a square.*