V. Association schemes

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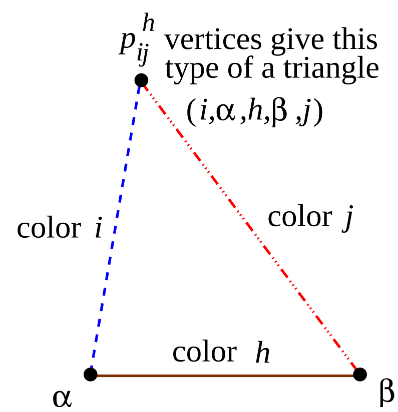

colored triangles over a fixed base

For a pair of given d-tuples $\boldsymbol{a}$ in $\boldsymbol{b}$ over an alphabet with $n \geq 2$ symbols, there are $d+1$ possible relations:

they can be equal, they can coincide on $d-1$ places, $d-2$ places, ..., or they can be distinct on all the places.

For a pair of given d-subsets A and B of the set with n elements, where $n \geq 2d$, we have $d+1$ possible relations:

they can be equal, they can intersect in $d-1$ elements, $d-2$ elements, ..., or they can be disjoint.

The above examples, together with the list of relations are examples of **association schemes** that we will introduce shortly.

In 1938 **Bose** and **Nair** introduced association schemes for applications in statistics.

However, it was **Philippe Delsarte** who showed in his thesis that association schemes can serve as a common framework for problems ranging from error-correcting codes, to combinatorial designs. Further connections include
– group theory (primitivity and imprimitivity),
– linear algebra (spectral theory),
– metric spaces,
– study of duality
– character theory,
– representation and orthogonal polynomials.

Bannai and Ito:

We can describe algebraic combinatorics as

"a study of combinatorial objects
with theory of characters"

or as

"a study of groups without a group".

Even more connections:

– knot theory (spin modules),
– linear programming bound,
– finite geometries.

A (symmetric) ***d*-class association scheme** on n vertices is a set of binary symmetric $(n \times n)$-matrices $I = A_0, A_1, \dots, A_d$ s.t.

(a) $\sum_{i=0}^{d} A_i = J$, where J is the all-one matrix,

(b) for all $i, j \in \{0, 1, \dots, d\}$ the product $A_i A_j$ is a linear combination of matrices $A_0, \dots, A_d$.

It is essentially a colouring of the edges of the complete graph K_n with d colours, such that the number of triangles with a given colouring on a given edge depends only on the colouring and not on the edge.

Bose-Mesner algebra

Subspace of $n \times n$ dim. matrices over $\mathbb{R}$ generated by $A_0, \ldots, A_d$ is, by (b), a *commutative algebra*, known as the **Bose-Mesner algebra** of $\mathcal{A}$ and denoted by $\boldsymbol{\mathcal{M}}$.

Since A_i is a symmetric binary matrix, it is the adjacency matrix of an (undirected) graph Γ_i on n vertices.

If the vertices x and y are connected in Γ_i, we will write $\boldsymbol{x \, \Gamma_i \, y}$ and say that they are in ***i*-th relation**.

The condition (a) implies that for every vertices x and y there exists a unique i, such that $x\,\Gamma_i\,y$, and that Γ_i, $i \neq 0$, has no loops.

The condition (b) implies that there exist such constants p_{ij}^h , $i, j, h \in \{0, \ldots, d\}$, that

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h. \qquad (3)$$

They are called **intersection numbers** of the association scheme $\mathcal{A}$. Since matrices A_i are symmetric, they commute. Thus also $p_{ij}^h = p_{ji}^h$.

By (3), the combinatorial meaning of intersection numbers p_{ij}^h, implies that they are integral and nonnegative.

Suppose $x\,\Gamma_h\,y$. Then

$$p_{ij}^h = |\{z\,;\; z\,\Gamma_i\,x \;\text{ in }\; z\,\Gamma_j\,y\}|. \tag{4}$$

Therefore, Γ_i is regular graph of valency $k_i := p_{ii}^0$ and we have $p_{ij}^0 = \delta_{ij}k_i$.

By counting in two different ways all triples (x, y, z), such that

$$x\,\Gamma_h\,y, \qquad z\,\Gamma_i\,x \qquad \text{and} \qquad z\,\Gamma_j\,y$$

we obtain also $k_h\,p_{ij}^h = k_j\,p_{ih}^j$.

Examples

Let us now consider some examples of associative schemes.

A scheme with one class consists of the identity matrix and the adjacency matrix of a graph, in which every two vertices are adjacent, i.e., a graph of diameter 1, i.e., the completer graph K_n.

We will call this scheme **trivial**.