ALGEBRAIC COMBINATORICS

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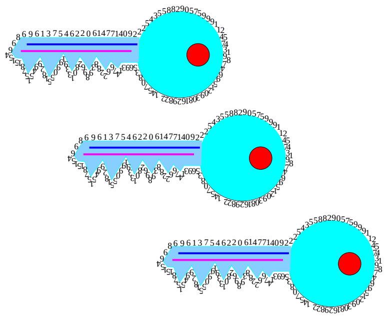


Introduction

We study an interplay between

algebra and **combinatorics**,

that is known under the name

algebraic combinatorics.

This is a discrete mathematics, where objects and structures contain some degree of regularity or symmetry.

More important areas of application of algebraic combinatorics are

- **coding theory** and error correction codes,
- **statistical design of experiments**, and
- (through finite geometries and finite fields) also **cryptography**.

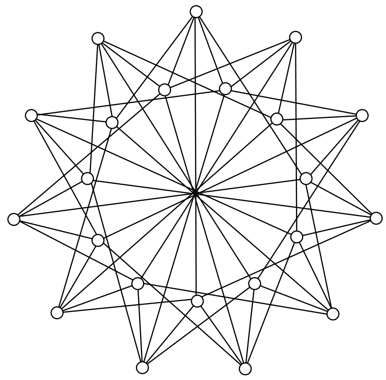
We investigate several interesting combinatorial structures. Our aim is a general introduction to algebraic combinatorics and illumination of some the important results in the past 10 years.

We will study as many topics as time permits that include:

- **algebraic graph theory and eigenvalue techniques** (specter of a graph and characteristic polynomial; equitable partitions: quotients and covers; strongly regular graphs and partial geometries, examples; distance-regular graphs, primitivity and classification, classical families),
- **associative schemes** (Bose-Mesner algebra, Krein conditions and absolute bounds; eigenmatrices and orthogonal relations, duality and formal duality, P-polynomial schemes, Q-polynomial schemes),
- **finite geometries and designs** (*projective and affine plane:* duality; *projective geometries:* spaces $\mathrm{PG}(d-1, q)$. *generalized quadrangles:* quadratic forms and a classification of isotropic spaces, classical constructions, small examples, spreads and regular points).

I. Constructions of some famous combinatorial objects

- Incidence structures
- Orthogonal Arrays (OA)
- Latin Squares (LS), MOLS
- Transversal Designs (TD)
- Hadamard matrices



Heawood's graph
the point/block incidence graph
of the unique 2-(11,5,2) design.

Incidence structures

t-(v, s, λ_t) design is

- a collection of s-subsets (**blocks**)
- of a set with v elements (**points**),
- where each t-subset of points is contained in exactly λ_t blocks.

If $\lambda_t = 1$, then the t-design is called **Steiner System** and is denoted by $S(t, s, v)$.

Let $i \in \mathbb{N}_0$, $i \leq t$ and let $\lambda_i(S)$ denotes the number of blocks containing a given i-set S. Then

(1) S is contained in $\lambda_i(S)$ blocks and each of them contains $\binom{s-i}{t-i}$ distinct t-sets with S as subset;

(2) the set S can be enlarged to t-set in $\binom{v-i}{t-i}$ ways and each of these t-set is contained in λ_t blocks:

$$\lambda_i(S)\binom{s-i}{t-i} = \lambda_t\binom{v-i}{t-i}$$

Therefore, $\lambda_i(S)$ is independent of S (so we can denote it simply by λ_i) and hence a t-design is also i-design, for $0 \leq i \leq t$.

For $\lambda_0 = b$ and $\lambda_1 = r$, when $t \geq 2$, we have

$$bs = rv \quad \text{and} \quad r(s-1) = \lambda_2(v-1)$$

or

$$r = \lambda_2 \frac{v-1}{s-1} \quad \text{and} \quad b = \lambda_2 \frac{v(v-1)}{s(s-1)}$$

Bruck-Ryser-Chowla Theorem

BRC Theorem (1963). *Suppose* $\exists \text{SBIBD}(v,k,\lambda)$.
If v *is even, then* $k-\lambda$ *is a square.*
If v *is odd, then the Diophantine equation*

$$x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$$

has nonzero solution in x, y *and* z.

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H.J. Ryser       M. Hall       Street\&Wallis  H.J. Ryser
Combin. math.    Combin. Th.   Combinatorics   paper
1963             1967          1982            (Witt cancellation law)
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All use Lagrange theorem: $m = a^2 + b^2 + c^2 + d^2$.

By the first part of BRC theorem there exists $\mathrm{PG}(2, n)$ for all $n \in \{2, \ldots, 9\}$, except possibly for $n = 6$.

The second part of BRC rules out this case, since there is no nontrivial solution of $z^2 = 6x^2 - y^2$.

For $n = 10$ the same approach fails for the first time, since the equation $z^2 = 10x^2 - y^2$ has a solution $(x, y, z) = (1, 1, 3)$.

Several hounderd hours on Cray 1 eventually ruled out this case.

A **partial linear space** is an incidence structure in which any two points are incident with at most one line. This implies that any two lines are incident with at most one point.

A **projective plane** is a partial linear space satisfying the following three conditions:

(1) Any two lines meet in a unique point.

(2) Any two points lie in a unique line.

(3) There are three pairvise noncolinear points (a *triangle*).

The **projective space** PG(d, n) (of dimension d and order q) is obtained from $[\mathrm{GF}(q)]^{d+1}$ by taking the quotient over linear spaces.

In particular, the projective space PG$(2, n)$ is the incidence structure with 1- and 2-dim. subspaces of $[\mathrm{GF}(q)]^3$ as **points** and **lines** (blocks), and **"being a subspace"** as an incidence relation.

PG$(2, n)$ is a 2-$(q^2 + q + 1, q + 1, 1)$-design, i.e.,

- $v = q^2 + q + 1$ is the number of points (and lines b),
- each line contains $k = q + 1$ points

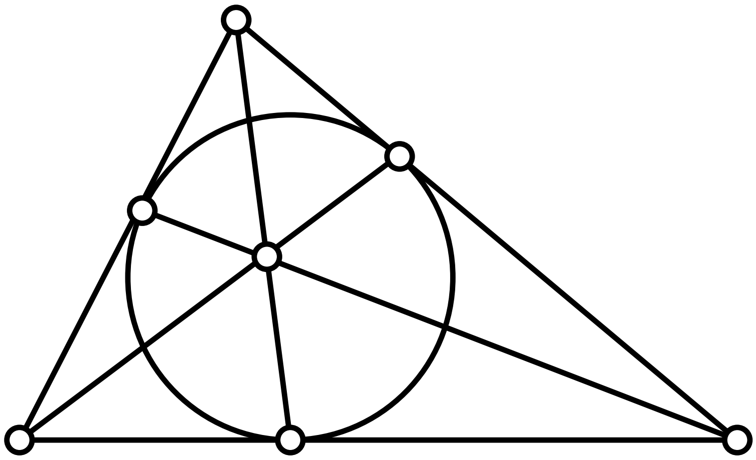
  (on each point we have $r = q + 1$ lines),
- each pair of points is on $\lambda = 1$ lines

  (each two lines intersect in a precisely one point),

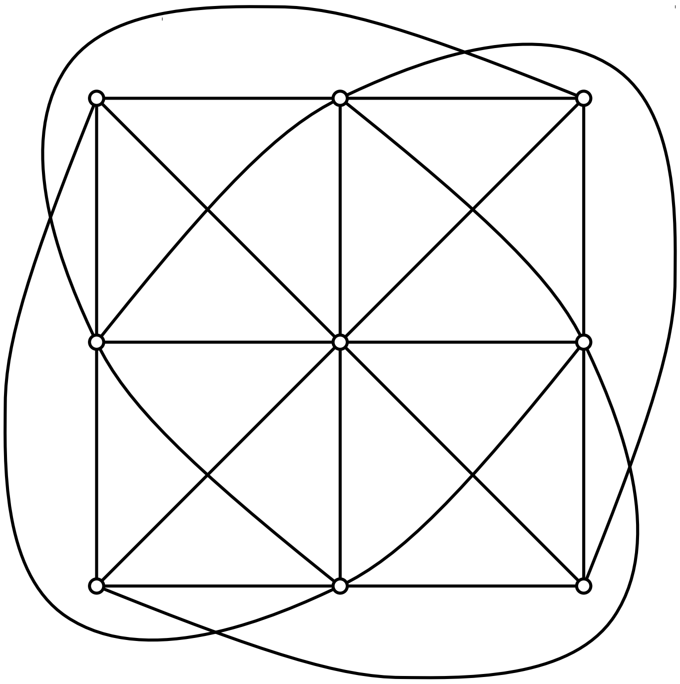
which is in turn a projective plane (see Assignment 1).

Examples:

1. The projective plane PG$(2,2)$ is also called the **Fano plane** (7 points and 7 lines).



2. PG$(2, 3)$ can be obtained from 3×3 grid (or AG$(2, 3)$).



3. PG$(2,4)$ is obtained from $\mathbb{Z}_{21}$:
points $= \mathbb{Z}_{21}$ and
lines $= \{S + x \mid x \in \mathbb{Z}_{21}\}$,
where S is a 5-element set $\{3, 6, 7, 12, 14\}$, i.e.,

$\{0, 3, 4, 9, 11\}$ $\{1, 4, 5, 10, 12\}$ $\{2, 5, 6, 11, 13\}$
$\{3, 6, 7, 12, 14\}$ $\{4, 7, 8, 13, 15\}$ $\{5, 8, 9, 14, 16\}$
$\{6, 9, 10, 15, 17\}$ $\{7, 10, 11, 16, 18\}$ $\{8, 11, 12, 17, 19\}$
$\{9, 12, 13, 18, 20\}$ $\{10, 13, 14, 19, 0\}$ $\{11, 14, 15, 20, 1\}$
$\{12, 15, 16, 0, 2\}$ $\{13, 16, 17, 1, 3\}$ $\{14, 17, 18, 2, 4\}$
$\{15, 18, 19, 3, 5\}$ $\{16, 19, 20, 4, 6\}$ $\{17, 20, 0, 5, 7\}$
$\{18, 0, 1, 6, 8\}$ $\{19, 1, 2, 7, 9\}$ $\{20, 2, 3, 8, 10\}$

Note: Similarly the Fano plane can be obtained from $\{0, 1, 3\}$ in $\mathbb{Z}_7$.

Let $\mathcal{O}$ be a subset of points of $\mathrm{PG}(2,n)$ such that no three are on the same line.

Then $|\mathcal{O}| \leq n+1$ if n is odd
and $|\mathcal{O}| \leq n+2$ if n is even.

If equality is attained then $\mathcal{O}$ is called
oval for n even, and **hyperoval** for n odd

Examples:

- the vertices of a triangle and the center of the circle in Fano plane,
- the vertices of a square in $\mathrm{PG}(2,3)$ form oval,
- the set of vertices $\{0,1,2,3,5,14\}$ in the above $\mathrm{PG}(2,4)$ is a hyperoval.

The **general linear group** $\mathrm{GL}_n(q)$ consists of all invertible $n \times n$ matrices with entries in $\mathrm{GF}(q)$.

The **special linear group** $\mathrm{SL}_n(q)$ is the subgroup of all matrices with determinant 1.

The **projective general linear group** $\mathrm{PGL}_n(q)$ and the **projective special linear group** $\mathrm{PSL}_n(q)$ are the groups obtained from $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$ by taking the quotient over scalar matrices (i.e., scalar multiple of the identity matrix).

For $n \geq 2$ the group $\mathrm{PSL}_n(q)$ is simple
(except for $\mathrm{PSL}_2(2) = S_3$ and $\mathrm{PSL}_2(3) = A_4$)
and is by Artin's convention denoted by $L_n(q)$.

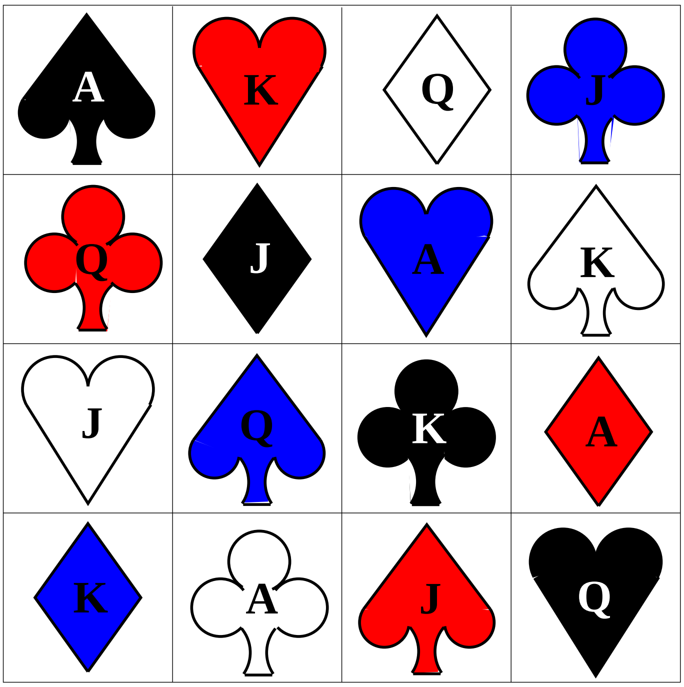
Orthogonal Arrays

An **orthogonal array**, OA(v, s, λ), is such $(\lambda v^2 \times s)$-dimensional matrix with v symbols, that each two columns each of v^2 possible pairs of symbols appears in exactly λ rows.

This and to them equivalent structures (e.g. transversal designs, pairwise orthogonal Latin squares, nets,...) are part of design theory.

If we use the first two columns of $\mathrm{OA}(v, s, 1)$ for coordinates, the third column gives us a **Latin square**, i.e., $(v \times v)$-dim. matrix in which all symbols $\{1, \dots, v\}$ appear in each row and each column.

Example : $\mathrm{OA}(3, 3, 1)$ $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$



Three pairwise orthogonal Latin squares of order 4, i.e., each pair symbol-letter or letter-color or color-symbol appears exactly once.

Theorem. *If* $\mathrm{OA}(v, s, \lambda)$ *exists, then we have in the case* $\lambda = 1$

$$s \leq v + 1,$$

and in general

$$\lambda \geq \frac{s(v-1)+1}{v^2}.$$

Transversal design $\mathrm{TD}_\lambda(s, v)$ is an incidence structure of blocks of size s, in which points are partitioned into s groups of size v so that an arbitrary points lie in λ blocks when they belong to distinct groups and there is no block containing them otherwise.

Proof: The number of all lines that intersect a chosen line of $\mathrm{TD}_1(s, v)$ is equal to $(v-1)s$ and is less or equal to the number of all lines without the chosen line, that is $v^2 - 1$.

In transversal design $\mathrm{TD}_\lambda(s, v)$, $\lambda \neq 1$ we count in a similar way and then use the inequality between arithmetic and quadratic mean (that can be derived from Jensen inequality). ■

Theorem. *For a prime p there exists* $\mathrm{OA}(p, p, 1)$, *and there also exists* $\mathrm{OA}(p, (p^d - 1)/(p - 1), p^{d-2})$ *for* $d \in \mathbb{N}\backslash\{1\}$

Proof: Set $\lambda = 1$. For $i, j, s \in \mathbb{Z}_p$ we define

$$e_{ij}(s) = is + j \bmod p.$$

For $\lambda \neq 1$ we can derive the existence from the construction of projective geometry $\mathrm{PG}(n, d)$. ■

For homework convince yourself that each $\mathrm{OA}(n, n, 1)$, $n \in \mathbb{N}$, can be extended for one more column, i.e., to $\mathrm{OA}(n, n + 1, 1)$.

Hadamard matrices

Let A be $n \times n$ matrix with $|a_{ij}| \leq 1$.

How large can $\det A$ be?

Since each column of A is a vector of length at most $\sqrt{n}$, we have

$$\det A \leq n^{n/2}.$$

Can equality hold? In this case all entries must be ± 1 and any two distinct columns must me orthogonal.

$(n \times n)$-dim. matrix H with elements ± 1, for which

$$HH^T = nI_n$$

holds is called a **Hadamard matrix** of order n.

Such a matrix exists only if $n = 1$, $n = 2$ or $4 \mid n$.

A famous **Hadamard matrix conjecture** (1893): a Hadamard matrix of order $4s$ exists $\forall s \in \mathbb{N}$.

In 2004 Iranian mathematicians H. Kharaghani and B. Tayfeh-Rezaie constructed a Hadamard matrix of order 428. The smallest open case is now 668.

$$n=2: \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad n=4: \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$n=8: \quad \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & - & - & - & + & - & + \\ + & + & + & - & - & - & + & - \\ + & - & + & + & - & - & - & + \\ + & + & - & + & + & - & - & - \\ + & - & + & - & + & + & - & - \\ + & - & - & + & - & + & + & - \\ + & - & - & - & + & - & + & + \end{pmatrix}$$

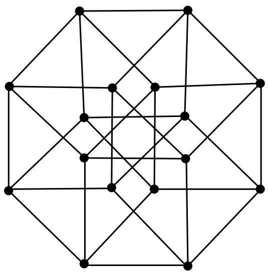
Hadamard matrix of order $4s$ is equivalent to 2-$(4s-1, 2s-1, s-1)$ design.

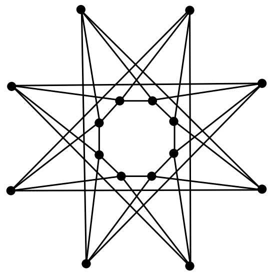
A recursive construction of a Hadamard matrix H_{nm} using H_n, H_m and Kronecker product (hint: use $(A \otimes B)(C \otimes D) = (AB) \otimes (BD)$ and $(A \otimes B)^t = A^t \otimes B^t$).

We could also use *conference matrices* (Belevitch 1950, use for teleconferencing) with 0 on the diagonal and $CC^t = (n-1)I$. in order to obtain two simple constructions: if C is antisymmetric ($H = I + C$) or symmetric (H_{2n} consists of four blocks of the form $\pm I \pm C$).

II. Graphs, eigenvalues and regularity

- adjacency matrix and walks,
- eigenvalues,
- regularity,
- eigenvalue multiplicities,
- Peron-Frobenious Theorem,
- interlacing.





the four-cube

$\boxed{\text{Graphs}}$

A **graph** Γ is a pair $(V\Gamma, E\Gamma)$, where $V\Gamma$ is a finite set of **vertices** and $E\Gamma$ is a set of unordered pairs xy of vertices called **edges** (no loops or multiple edges).

Let $V\Gamma = \{1, \ldots, n\}$. Then a $(n \times n)$-dim. matrix A is the **adjacency matrix** of Γ, when

$$A_{i,j} = \begin{cases} 1, & \text{if } \{i, j\} \in E, \\ 0, & \text{otherwise} \end{cases}$$

$\boxed{\textbf{Lemma.} \quad (A^h)_{ij} = \#\ \textit{walks from } i \textit{ to } j \textit{ of length } h.}$

Eigenvalues

The number $\theta \in \mathbb{R}$ is an **eigenvalue** of Γ, when for a vector $x \in \mathbb{R}^n \backslash \{0\}$ we have

$$Ax = \theta x, \qquad \text{i.e.,} \qquad (Ax)_i = \sum_{\{j,i\} \in E} x_j = \theta x_i.$$

- There are cospectral graphs, e.g. $K_{1,4}$ and $K_1 \cup C_4$.
- A triangle inequality implies that the maximum degree of a graph Γ, denoted by $\Delta(\Gamma)$, is greater or equal to $|\theta|$, i.e.,

$$\Delta(\Gamma) \geq |\theta|.$$

A graph with **precisely one** eigenvalue is a graph with one vertex, i.e., a graph with diameter **0**.

A graph with **two** eigenvalues is the complete graph K_n, $n \geq 2$, i.e., the graph with diameter **1**.

Theorem. *A connected graph of diameter d has at least $d+1$ distinct eigenvalues.*

Review of basic matrix theory

Lemma. *Let A be a real symetric matrix. Then*

- *its eigenvalues are real numbers, and*
- *the eigenvectors corresponding to distinct eigenvalues, then they are orthogonal.*
- *If U is an A-invariant subspace of $\mathbb{R}^n$, then $U^\perp$ is also A-invariant.*
- *$\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of A.*
- *There are matrices L and D, such that*

$$L^T L = LL^T = I \quad \text{and} \quad LAL^T = D,$$

where D is a diagonal matrix of eigenvalues of A.

Lemma. *The eigenvalues of a disconnected graph are just the eigenvalues of its components and their multiplicities are sums of the corresponding multiplicities in each component.*

Regularity

A graph is **regular**, if each vertex has the same number of neighbours.

Set $\boldsymbol{j}$ to the be all-one vector in $\mathbb{R}^n$.

Lemma. *A graph is regular iff $\boldsymbol{j}$ is its eigenvector.*

Lemma. *If Γ is a regular graph of valency k, then the multiplicity of k is equal to the number of connected components of Γ,*
and the multiplicity of $-k$ is equal to the number of bipartite components of Γ.

Lemma. *Let Γ be a k-regular graph on n vertices with eigenvalues $k, \theta_2, \ldots, \theta_n$. Then Γ and $\overline{\Gamma}$ have the same eigenvectors, and the eigenvalues of $\overline{\Gamma}$ are $n-k-1$, $-1-\theta_2, \ldots, -1-\theta_n$.*

Calculate the eigenvalues of many simple graphs:

- $m * K_n$ and their complements,
- circulant graphs
- C_n,
- $K_n \times K_n$,
- Hamming graphs,...

Line graphs and their eigenvalues

We call $\boldsymbol{\phi(\Gamma, x)} = \det(xI - A(\Gamma))$
the **characteristics polynomial** of a graph Γ.

Lemma. *Let B be the incidence matrix of the graph Γ, L its line graph and $\Delta(\Gamma)$ the diagonal matrix of valencies. Then*

$$B^T B = 2I + A(L) \quad \text{and} \quad BB^T = \Delta(\Gamma) + A(\Gamma).$$

Furthermore, if Γ is k-regular, then

$$\phi(L, x) = (x+2)^{e-n}\phi(\Gamma, x-k+2).$$

Semidefinitness

A real symmetric matrix A is **positive semidefinite** if

$$u^T Au \geq 0 \qquad \text{for all vectors } u.$$

It is **positive definite** if it is positive semidefinite and

$$u^T Au = 0 \iff u = 0.$$

Characterizations.

- *A positive semidefinite matrix is positive definite iff invertible*
- *A matrix is positive semidefinite matrix iff all its eigenvalues are nonnegative.*
- *If $A = B^T B$ for some matrix, then A is positive semidefinite.*

The **Gram matrix** of vectors $u_1, \dots, u_n \in \mathbb{R}^m$ is $n \times n$ matrix G s.t. $G_{ij} = u_i^t u_j$.

Note that $B^T B$ is the Gram matrix of the columns of B, and that any Gram matrix is positive semidefinite. The converse is also true.

Corollary. *The least eigenvalue of a line graph is at least* -2. *If* Δ *is an induced subgraph of* Γ, *then*

$$\theta_{\min}(\Gamma) \leq \theta_{\min}(\Delta) \leq \theta_{\max}(\Delta) \leq \theta_{\max}(\Gamma).$$

Let $\boldsymbol{\rho(A)}$ be the **spectral radious** of a matrix A.

Peron-Frobenious Theorem. *Suppose A is a nonnegative $n \times n$ matrix, whose underlying directed graph X is strongly connected. Then*

(a) *$\rho(A)$ is a simple eigenvalue of A. If x an eigenvector for ρ, then no entries of x are zero, and all have the same sign.*

(b) *Suppose A_1 is a real nonnegative $n \times n$ matrix such that $A - A_1$ is nonnegative.*
Then $\rho(A_1) \leq \rho(A)$, with equality iff $A_1 = A$.

(c) *If θ is an eigenvalue of A and $|\theta| = \rho(A)$, then $\theta/\rho(A)$ is an mth root of unity and $e^{2\pi ir/m}\rho(A)$ is an eigenvalue of A for all r. Furthermore, all cycles in X have length divisible by m.*

Let A be a symmetric $n \times n$ matrix and let us define a real-valued function f on $\mathbb{R}^n$ by

$$f(\boldsymbol{x}) := \frac{(\boldsymbol{x}, A\boldsymbol{x})}{(\boldsymbol{x}, \boldsymbol{x})}.$$

Let $\boldsymbol{x}$ and $\boldsymbol{u}$ be orthogonal unit vectors in $\mathbb{R}^n$ and set $\boldsymbol{x}(\varepsilon) := \boldsymbol{x} + \varepsilon\boldsymbol{u}$. Then $\big(\boldsymbol{x}(\varepsilon), \boldsymbol{x}(\varepsilon)\big) = 1 + \varepsilon^2$,

$$f(\boldsymbol{x}(\varepsilon)) = \frac{(\boldsymbol{x}, A\boldsymbol{x}) + 2\varepsilon(\boldsymbol{u}, A\boldsymbol{x}) + \varepsilon^2(\boldsymbol{u}, A\boldsymbol{u})}{1 + \varepsilon^2}$$

and

$$\lim_{\varepsilon \to 0} \frac{f(\boldsymbol{x}(\varepsilon)) - f(\boldsymbol{x})}{\varepsilon} = 2(\boldsymbol{u}, A\boldsymbol{x}).$$

So f has a local extreme iff $(\boldsymbol{u}, A\boldsymbol{x}) = 0\ \forall \boldsymbol{u} \perp \boldsymbol{x}$ and $||\boldsymbol{u}|| = 1$ iff for every $\boldsymbol{u} \perp \boldsymbol{x}$ we have $\boldsymbol{u} \perp A\boldsymbol{x}$ iff $A\boldsymbol{x} = \theta\boldsymbol{x}$ for some $\theta \in \mathbb{R}$. More precisely:

Theorem [Courant-Fischer].
Let A be a symmetric $n \times n$ matrix with eigenvalues $\theta_1 \geq \cdots \geq \theta_n$. Then

$$\theta_k = \max_{\dim(U)=k} \min_{x \in U} \frac{(\boldsymbol{x}, A\boldsymbol{x})}{(\boldsymbol{x}, \boldsymbol{x})} = \min_{\dim(U)=n-k+1} \max_{x \in U} \frac{(\boldsymbol{x}, A\boldsymbol{x})}{(\boldsymbol{x}, \boldsymbol{x})}.$$

Using this result, it is not difficult to prove the following (generalized) interlacing result.

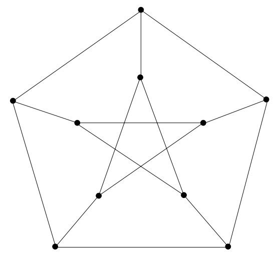
Theorem [Haemers]. *Let A be a complete hermitian $n \times n$ matrix, partitioned into m^2 block matrices, such that all diagonal matrices are square. Let B be the $m \times m$ matrix, whose i, j-th entry equals the average row sum of the i, j-th block matrix of A for $i, j = 1, \dots, m$. Then the eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_m$ of A and B resp. satisfy*

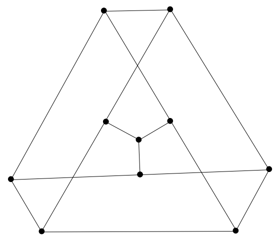
$$\boldsymbol{\alpha_i \geq \beta_i \geq \alpha_{i+n-m}, \quad for \quad i = 1, \dots, m.}$$

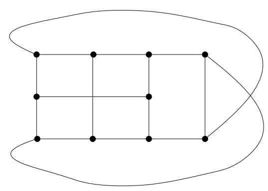
Moreover, if for some $k \in N_0$, $k \leq m$, $\alpha_i = \beta_i$ for $i = 1, \dots, k$ and $\beta_i = \alpha_{i+n-m}$ for $i = k+1, \dots, m$, then all the block matrices of A have constant row and column sums.

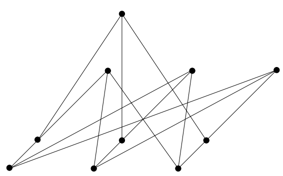
III. Strongly regular graphs

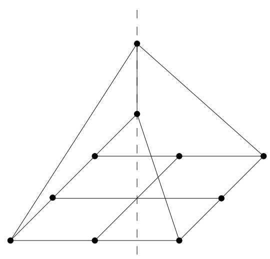
- definition of strongly regular graphs
- characterization with adjacency matrix
- classification (type I in II)
- Paley graphs
- Krein condition and Smith graphs
- more examples (Steiner and LS graphs)
- feasibility conditions and a table











Definition

Two similar regularity conditions are:

(a) any two adjacent vertices have exactly λ common neighbours,

(b) any two nonadjacent vertices have exactly μ common neighbours.

A regular graph is called **strongly regular** when it satisfies (a) and (b). Notation $\mathbf{SRG}(\boldsymbol{n}, \boldsymbol{k}, \boldsymbol{\lambda}, \boldsymbol{\mu})$, where k is the valency of Γ and $n = |V\Gamma|$.

Strongly regular graphs can also be treated as extremal graphs and have been studied extensively.

Examples

5-cycle is $\mathrm{SRG}(5,2,0,1)$,
the Petersen graph is $\mathrm{SRG}(10,3,0,1)$.

What are the trivial examples?

$K_n, \qquad m \cdot K_n,$

The **Cocktail Party graph $C(n)$**, i.e., the graph on $2n$ vertices of degree $2n-2$, is also strongly regular.

Lemma. *A strongly regular graph* Γ *is disconnected iff* $\mu = 0$.

If $\mu = 0$*, then each component of* Γ *is isomorphic to* K_{k+1} *and we have* $\lambda = k - 1$.

Corollary. *A complete multipartite graph is strongly regular iff its complement is a union of complete graphs of equal size.*

Homework: Determine all SRG with $\mu = k$.

Counting the edges between the neighbours and non-neighbours of a vertex in a connected strongly regular graph we obtain:

$$\mu(n-1-k) = k(k-\lambda-1),$$

i.e.,

$$n = 1 + k + \frac{k(k-\lambda-1)}{\mu}.$$

Lemma. *The complement of* $\mathrm{SRG}(n,k,\lambda,\mu)$ *is again strongly regular graph:*

$$\mathrm{SRG}(\overline{n},\overline{k},\overline{\lambda},\overline{\mu}) = (n, n-k-1, n-2k+\mu-2, n-2k+\lambda).$$

Let J be the all-one matrix of dim. $(n \times n)$.

A graph Γ on n vertices is strongly regular if and only if its adjacency matrix A satisfies

$$A^2 = kI + \lambda A + \mu(J - I - A),$$

for some integers k, λ and μ.

Therefore, the valency k is an eigenvalue with multiplicity 1 and the nontrivial eigenvalues, denoted by σ and τ, are the roots of

$$x^2 - (\lambda - \mu)x + (\mu - k) = 0,$$

and hence $\lambda - \mu = \sigma + \tau$, $\mu - k = \sigma\tau$.

Theorem. *A connected regular graph with precisely three eigenvalues is strongly regular.*

Proof. Consider the following matrix polynomial:

$$M := \frac{(A-\sigma)(A-\tau)}{(k-\sigma)(k-\tau)}$$

If $A = A(\Gamma)$, where Γ is a connected k-regular graph with eigenvalues k, σ and τ, then all the eigenvalues of M are 0 or 1. But all the eigenvectors corresponding to σ and τ lie in $\mathrm{Ker}(A)$, so $\mathrm{rank} M{=}1$ and $M\boldsymbol{j} = \boldsymbol{j}$,

hence $M = \dfrac{1}{n}J$. and $A^2 \in \mathrm{span}\{I, J, A\}$. ■

For a connected graph, i.e., $\mu \neq 0$, we have

$$n = \frac{(k-\sigma)(k-\tau)}{k+\sigma\tau}, \quad \lambda = k+\sigma+\tau+\sigma\tau, \quad \mu = k+\sigma\tau$$

and the multiplicities of σ and τ are

$$m_\sigma = \frac{(n-1)\tau + k}{\tau - \sigma} = \frac{(\tau+1)k(k-\tau)}{\mu(\tau-\sigma)}$$

and $m_\tau = n - 1 - m_\sigma$.

Multiplicities

Solve the system:

$$\begin{aligned} 1 + m_\sigma + m_\tau &= n \\ 1 \cdot k + m_\sigma \cdot \sigma + m_\tau \cdot \tau &= 0. \end{aligned}$$

to obtain

$$m_\sigma \text{ and } m_\tau = \frac{1}{2}\left(n - 1 \pm \frac{(n-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right).$$

Classification

We classify strongly regular graphs into two types:

Type I (or conference) graphs: for these graphs $(n-1)(\mu-\lambda) = 2k$, which implies $\lambda = \mu-1$, $k = 2\mu$ and $n = 4\mu+1$, i.e., the strongly regular graphs with the same parameters as their complements.

They exist iff n is the sum of two squares.

Type II graphs: for these graphs $(\mu-\lambda)^2+4(k-\mu)$ is a perfect square Δ^2, where Δ divides $(n-1)(\mu-\lambda)-2k$ and the quotient is congruent to $n-1 \pmod 2$.

Paley graphs

q a prime power, $q \equiv 1 \pmod 4$ and set $\mathbb{F} = \mathrm{GF}(q)$. The **Paley graph** $\boldsymbol{P(q)}$$= (V, E)$ is defined by:

$$V = \mathbb{F} \quad \text{and} \quad E = \{(a,b) \in \mathbb{F} \times \mathbb{F} \,|\, (a-b) \in (\mathbb{F}^*)^2\}.$$

i.e., two vertices are adjacent if their difference is a non-zero square. $P(q)$ is **undirected**, since $-1 \in (\mathbb{F}^*)^2$.

Consider the map $x \longrightarrow x + a$, where $a \in \mathbb{F}$, and the map $x \longrightarrow xb$, where $b \in \mathbb{F}$ is a square or a nonsquare, to show $P(q)$ is **strongly regular** with

$$\text{valency } k = \frac{q-1}{2}, \quad \lambda = \frac{q-5}{4} \quad \text{and} \quad \mu = \frac{q-1}{4}.$$

Seidel showed that these graphs are uniquely determined with their parameters for $q \leq 17$.

There are some results in the literature showing that Paley graphs behave in many ways like random graphs $G(n, 1/2)$.

Bollobás and Thomason proved that the Paley graphs contain all small graphs as induced subgraphs.

Krein conditions

Of the other conditions satisfied by the parameteres of SRG, the most important are the **Krein conditions**, first proved by Scott using a result of Krein from harmonic analysis:

$$(\sigma+1)(k+\sigma+2\sigma\tau) \leq (k+\sigma)(\tau+1)^2$$

and

$$(\tau+1)(k+\tau+2\sigma\tau) \leq (k+\tau)(\sigma+1)^2.$$

Some parameter sets satisfy all known necessary conditions. We will mention some of these.

If $k > s > t$ eigenvalues of a strongly regular graph, then the first inequality translates to

$$k \geq -s\,\frac{(2t+1)(t-s)-t(t+1)}{(t-s)+t(t+1)},$$

$$\lambda \geq -(s+1)t\,\frac{(t-s)-t(t+3)}{(t-s)+t(t+1)},$$

$$\mu \geq -s(t+1)\,\frac{(t-s)-t(t+1)}{(t-s)+t(t+1)},$$

A strongly regular graph with parameters (k, λ, μ) given by taking equalities above, where t and s are integers such that $t - s \geq t(t+3)$ (i.e., $\lambda \geq 0$) and $k > t > s$ is called a **Smith graph**.

A strongly regular graph with eigenvalues $k > \sigma > \tau$ is said to be of **(negative) Latin square type** when $\mu = \tau(\tau + 1)$ (resp. $\mu = \sigma(\sigma + 1)$).

The complement of a graph of (negative) Latin square type is again of (negative) Latin square type.

A graph of Latin square type is denoted by $\mathrm{L}_u(v)$, where $u = -\sigma$, $v = \tau - \sigma$ and it has the same parameters as the line graph of a $\mathrm{TD}_u(v)$.

Graphs of negative Latin square type ware introduced by Mesner, and are denoted by $\mathrm{NL}_e(f)$, where $e = \tau$, $f = \tau - \sigma$ and its parameters can be obtained from $\mathrm{L}_u(v)$ by replacing u by $-e$ and v by $-f$.

More examples of strongly regular graphs:

$L(K_v)$ is strongly regular with parameters

$$n = \binom{v}{2}, \quad k = 2(v-1), \quad \lambda = v-2, \quad \mu = 4.$$

For $v \neq 8$ this is the unique srg with these parameters.

Similarly, $L(K_{v,v}) = K_v \times K_v$ is strongly regular, with parameters

$$n = v^2, \quad k = 2(v-2), \quad \lambda = v-2, \quad \mu = 2.$$

and eigenvalues $\quad 2(v-1)^1, \quad v-2^{2(v-1)}, \quad -2^{(v-1)^2}.$

For $v \neq 4$ this is the unique srg with these parameters.

Steiner graph is the block (line) graph of a 2-$(v, s, 1)$ design with $v-1 > s(s-1)$, and it is strongly regular with parameters

$$n = \frac{\binom{v}{2}}{\binom{s}{2}}, \qquad k = s\Big(\frac{v-1}{s-1} - 1\Big),$$

$$\lambda = \frac{v-1}{s-1} - 2 + (s-1)^2, \qquad \mu = s^2.$$

and eigenvalues

$$k^1, \quad \left(\frac{v-s^2}{s-1}\right)^{v-1}, \quad -s^{n-v}.$$

When in a design $\mathcal{D}$ the block size is two, the number of edges of the point graph equals the number of blocks of the design $\mathcal{D}$. In this case the line graph of the design $\mathcal{D}$ is the line graph of the point graph of $\mathcal{D}$. This justifies the name: the line graph of a graph.

A point graph of a Steiner system is a complete graph, thus a line graph of a Steiner system $S(2, v)$ is the line graph of a complete graph K_v, also called the **triangular graph**.

(If $\mathcal{D}$ is a square design, i.e., $v - 1 = s(s - 1)$, then its line graph is the complete graph K_v.)

The fact that Steiner triple system with v points exists for all $v \equiv 1$ or $3 \pmod 6$ goes back to Kirkman in 1847. More recently Wilson showed that the number $n(v)$ of Steiner triple systems on an andmissible number v of points satisfies

$$n(v) \geq \exp(v^2 \log v/6 - cv^2).$$

A Steiner triple system of order $v > 15$ can be recovered uniquely from its line graph, hence there are super-exponentially many $\mathrm{SRG}(n, 3s, s+3, 9)$, for $n = (s+1)(2s+3)/3$ and $s \equiv 0$ or $2 \pmod 3$.

For $2 \le s \le v$ the **block graph** of a transversal design $\mathrm{TD}(s, v)$ (two blocks being adjacent iff they intersect) is strongly regular with parameters $n = v^2$,

$$k = s(v-1), \ \ \lambda = (v-2)+(s-1)(s-2), \ \ \mu = s(s-1).$$

and eigenvalues

$$s(v-1)^1, \quad v-s^{s(v-1)}, \quad -s^{(v-1)(v-s+1)}.$$

Note that a line graph of $\mathrm{TD}(s, v)$ is a conference graph when $v = 2s-1$. For $s = 2$ we get **the lattice graph** $K_v \times K_v$.

The number of Latin squares of order k is asymptotically equal to

$$\exp(k^2 \log k - 2k^2)$$

Theorem *(Neumaier). The strongly regular graph with the smallest eigenvalue $-m$, $m \geq 2$ integral, is with finitely many exceptions, either a complete multipartite graph, a Steiner graph, or the line graph of a transversal design.*

Feasibility conditions and a table

- divisibility conditions
- integrality of eigenvalues
- integrality of multiplicities
- Krein conditions
- Absolute bounds

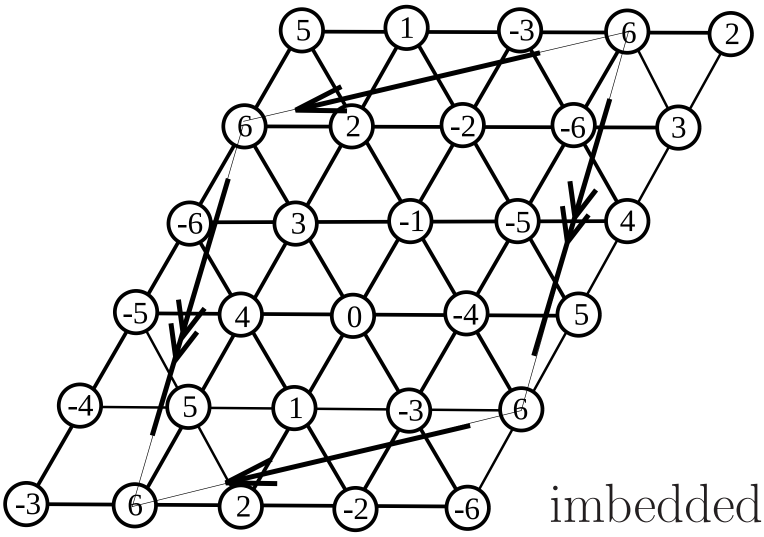
$$n \leq \frac{1}{2} m_\sigma (m_\sigma + 3),$$

and if $q_{11}^1 \neq 0$ even

$$n \leq \frac{1}{2} m_\sigma (m_\sigma + 1).$$

	n	k	λ	μ	σ	τ	m_σ	m_τ	graph
!	5	2	0	1	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	2	2	$C_5 = P(5)$ - Seidel
!	9	4	1	2	1	-2	4	4	$C_3 \times C_3 = P(9)$
!	10	3	0	1	1	-2	5	4	Petersen=compl. $T(5)$
!	13	6	2	3	$\frac{-1+\sqrt{13}}{2}$	$\frac{-1-\sqrt{13}}{2}$	6	6	$P(13)$
!	15	6	1	3	1	-3	9	5	GQ(2,2)=compl. $T(6)$
!	16	5	0	2	1	-3	10	5	Clebsch
2!	16	6	2	2	2	-2	6	9	Shrikhande, $K_4 \times K_4$
!	17	8	3	4	$\frac{-1+\sqrt{17}}{2}$	$\frac{-1-\sqrt{17}}{2}$	8	8	$P(17)$
!	21	10	3	6	1	-4	14	6	compl. T(7)
0	21	10	4	5	$\frac{-1+\sqrt{21}}{2}$	$\frac{-1-\sqrt{21}}{2}$	10	10	conference
!	25	8	3	2	3	-2	8	16	$K_5 \times K_5$
15!	25	12	5	6	2	-3	12	12	$P(25)$ (Paulus)
10!	26	10	3	4	2	-3	12	13	(Paulus)
!	27	10	1	5	1	-5	20	6	GQ(2,4)=compl. Schlaefli
4!	28	12	6	4	4	-2	7	20	$T(8)$ (Chang)
41!	29	14	6	7	$\frac{-1+\sqrt{29}}{2}$	$\frac{-1-\sqrt{29}}{2}$	14	14	$P(29)$, (Bussemaker & Spence)

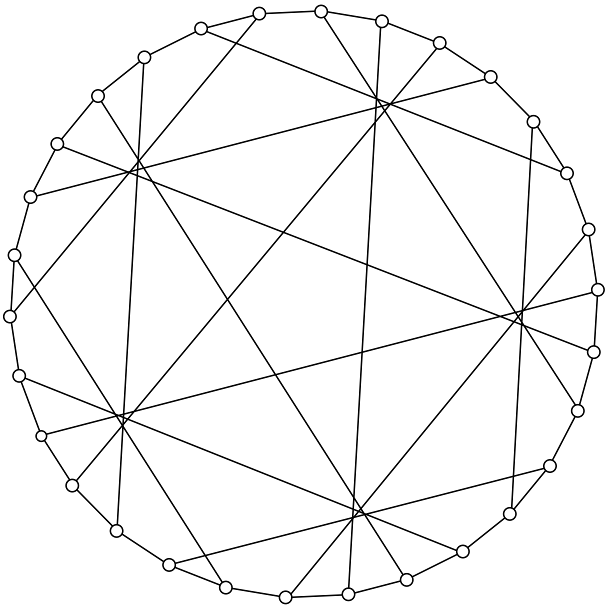
Paley graph $P(13)$



imbedded on a torus

The Shrikhande graph and $P(13)$ are the only distance-regular graphs which are locally C_6 (one has $\mu = 2$ and the other $\mu = 3$).

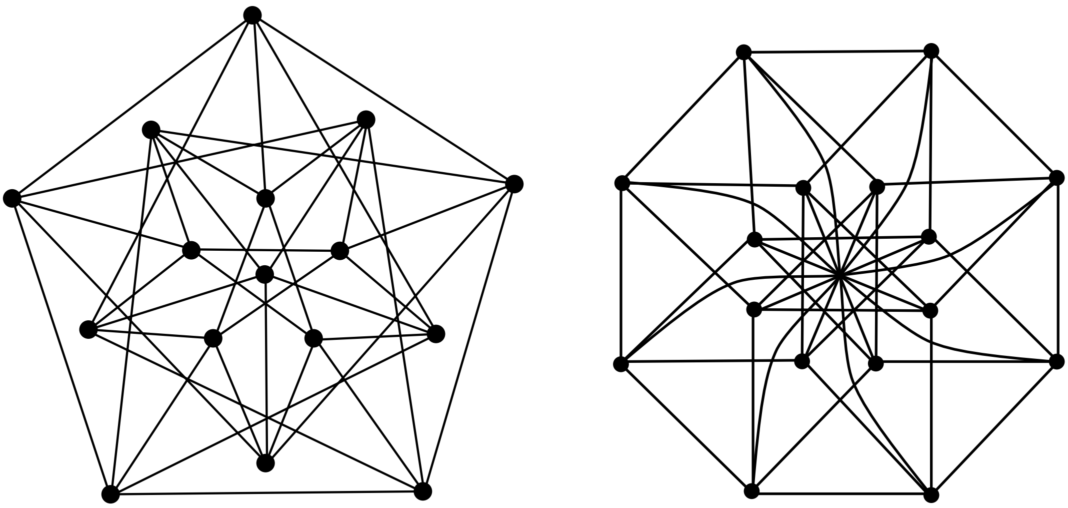
Tutte 8-cage



The Tutte's 8-cage is the $\mathrm{GQ}(2,2) = W(2)$.

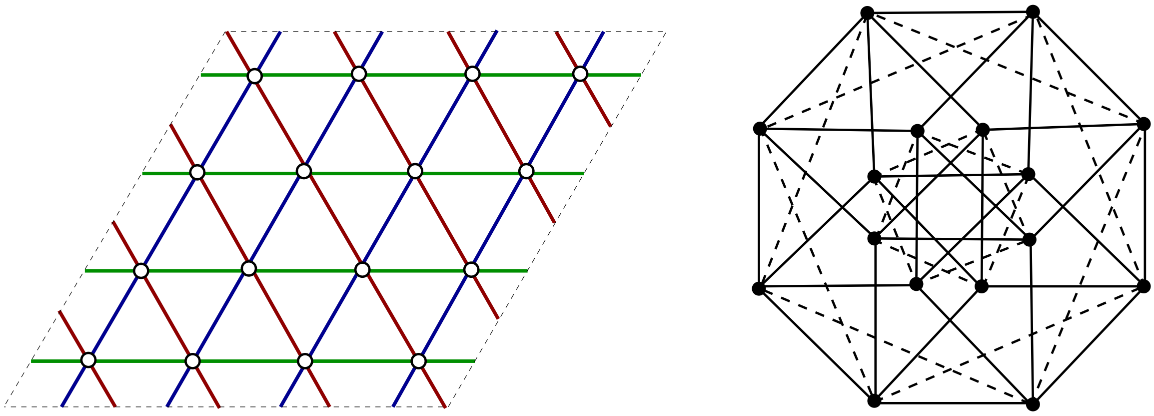
A **cage** is the smallest possible regular graph (here degree 3) that has a prescribed girth.

Clebsch graph



Two drawings of the complement of the Clebsch graph.

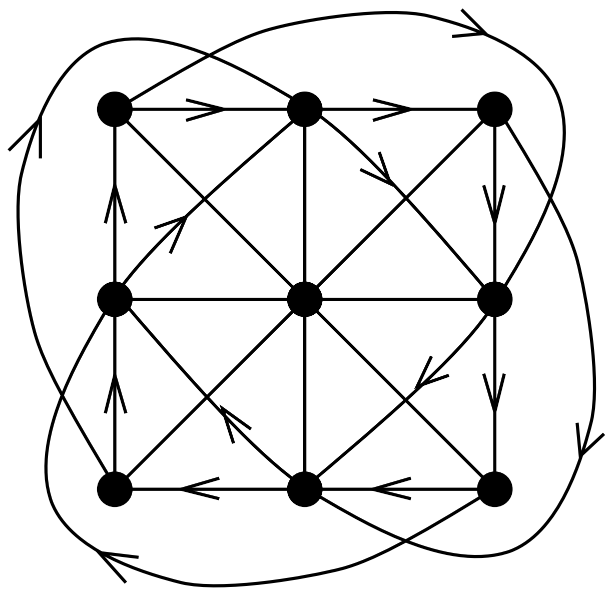
Shrikhande graph



The Shrikhande graph drawn on two ways:
(a) on a torus, (b) with imbedded four-cube.

The Shrikhande graph is not distance transitive, since some μ-graphs, i.e., the graphs induced by common neighbours of two vertices at distance two, are K_2 and some are $2{\cdot}K_1$.

Schläfly graph



How to construct the Schläfli graph:
make a cyclic 3-cover corresponding to arrows,
and then join vertices in every antipodal class.

Let Γ be a graph of diameter $\boldsymbol{d}$.

Then Γ has girth at most $\mathbf{2}\boldsymbol{d}+\mathbf{1}$,

while in the bipartite case the girth is at most $\mathbf{2}\boldsymbol{d}$.

Graphs with diameter d and girth $2d+1$ are called **Moore graphs** (**Hoffman and Singleton**).

Bipartite graphs with diameter d and girth $2d$ are known as **generalized polygons** (**Tits**).

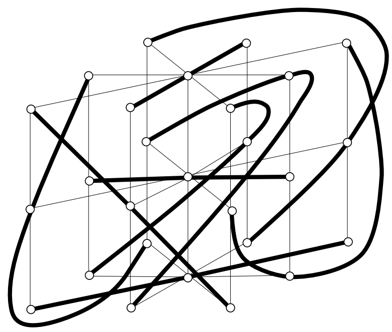
A **Moore graph of diameter two** is a regular graph with girth five and diameter two.

The only Moore graphs are

- the pentagon,
- the Petersen graph,
- the Hoffman-Singleton graph, and
- possibly a strongly regular graph on 3250 vertices.

IV. Geometry

- partial geometries
- classfication
- pseudogeometric
- quadratic forms
- isotropic spaces
- classical generalized quadrangles
- small examples



A unique spread

in GQ(3,3)=W(3)

A triple (P, L, I), i.e., (points,lines,incidence), is called a **partial geometry** $pg(R, K, T)$, when $\forall \ell, \ell' \in L$, $\forall p, p' \in P$:

- $|\ell| = K$, $\;|\ell \cap \ell'| \leq 1$,
- $|p| = R$, $\quad$ at most one line on p and p',
- if $p \notin \ell$, then there are exactly T points on ℓ that are collinear with p.

The **dual** (L, P, I^t) of a $pg(R, K, T)$ is again a partial geometry, with parameters (K, R, T).

Classification

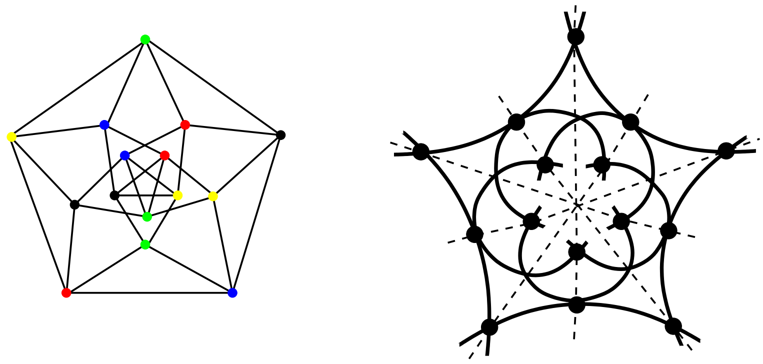
We divide partial geometries into four classes:

1. $T = K$: 2-$(v, K, 1)$ design,
2. $T = R - 1$: net,

   $T = K - 1$: transversal design,
3. $T = 1$: a generalized quadrangle $\mathrm{GQ}(K-1, R-1)$,
4. For $1 < T < \min\{K-1, R-1\}$ we say we have a **proper partial geometry**.

A $pg(t+1, s+1, 1)$ is a **generalized quadrangle** $\mathrm{GQ}(s, t)$.

An example



L(Petersen) is the point graph of the GQ$(2, 2)$ minus a spread (where spread consists of antipodal classes).

What about trivial examples?

Pseudo-geometric

The **point graph** of a $pg(P, L, I)$ is the graph with vertex set $X = P$ whose edges are the pairs of collinear points (also known as the *collinearity graph*).

The point graph of a $pg(R, K, T)$ is SRG:
$k = R(K-1)$, $\lambda = (R-1)(T-1)+K-2$, $\mu = RT$,
and eigenvalues $r = K - 1 - T$ and $s = -R$.

A SRG is called **pseudo-geometric** (R, K, T) if its parameters are as above.

Quadratic forms

A **quadratic form $\boldsymbol{Q(x_0, x_1, \ldots, x_n)}$** over $\mathrm{GF}(q)$ is a homogeneous polynomial of degree 2,

i.e., for $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ and an $(n+1)$-dim square matrix C over $\mathrm{GF}(q)$:

$$Q(\boldsymbol{x}) = \sum_{i,j=0}^{n} c_{ij} x_i x_j = \boldsymbol{x} C \boldsymbol{x}^T.$$

A **quadric** in $\mathrm{PG}(n, q)$ is the set of **isotropic** points:

$$Q = \{\langle \boldsymbol{x} \rangle \mid Q(\boldsymbol{x}) = 0\},$$

where $\langle \boldsymbol{x} \rangle$ is the 1-dim. subspace of $\mathrm{GF}(q)^{n+1}$ generated by $\boldsymbol{x} \in (\mathrm{GF}(q))^{n+1}$.

Two quadratic forms $Q_1(\boldsymbol{x})$ and $Q_2(\boldsymbol{x})$ are **projectively equivalent** if there is an invertible matrix A and $\boldsymbol{\lambda} \neq 0$ such that

$$Q_2(\boldsymbol{x}) = \boldsymbol{\lambda} Q_1(\boldsymbol{x}A).$$

The **rank** of a quadratic form is the smallest number of indeterminates that occur in a projectively equivalent quadratic form.

A quadratic form $Q(x_0, \dots, x_n)$ (or the quadric Q in $\mathrm{PG}(n, q)$ determined by it) is **nondegenerate** if its rank is $n+1$. (i.e., $Q \cap Q^{\perp} = 0$ and also to $Q^{\perp} = 0$).

For q odd a subspace U is **degenerate** whenever

$$U \cap U^{\perp} \neq \emptyset,$$

i.e., whenever its orthogonal complement $U^{\perp}$ is degenerate, where $\perp$ denotes the inner product on the vector space $V(n+1,q)$ defined by

$$(x,y) := Q(x+y) - Q(x) - Q(y).$$

Isotropic spaces

A **flat** of projective space $\mathrm{PG}(n, q)$
(defined over $(n+1)$-dim. space V)
consists of 1-dim. subspaces of V
that are contained in some subspace of V.

A flat is said to be **isotropic** when all its points are isotropic.

The dimension of maximal isotropic flats will be determined soon.

Theorem. *A nondegenerate quadric $Q(\boldsymbol{x})$ in* $\mathrm{PG}(n,q)$*, q odd, has the following canonical form*

(i) *for n even:* $Q(\boldsymbol{x}) = \sum_{i=0}^{n} x_i^2$,

(ii) *for n odd:*

(a) $Q(\boldsymbol{x}) = \sum_{i=0}^{n} x_i^2$,

(b) $Q(\boldsymbol{x}) = \eta x_0^2 + \sum_{i=1}^{n} x_i^2$, *where η is not a square.*

Theorem. *Any nondegenerate quadratic form $Q(\boldsymbol{x})$ over $\mathrm{GF}(q)$ is projectively equivalent to*

(i) *for* $n=2s$: $\mathcal{P}_{2s} = x_0^2+\sum_{i=1}^{s} x_{2i}x_{2i-1}$ (**parabolic**),

(ii) *for* $n = 2s-1$

(a) $\mathcal{H}_{2s-1} = \sum_{i=0}^{s-1} x_{2i}x_{2i+1}$ (**hyperbolic**),

(b) $\mathcal{H}_{2s-1} = \sum_{i=1}^{s-1} x_{2i}x_{2i+1} + f(x_0, x_1)$, (**elliptic**)

where f is an irreducible quadratic form.

The dimension of maximal isotropic flats:

Theorem. *A nondegenerate quadric Q in* $\mathrm{PG}(n,q)$ *has the following number of points and maximal projective dim. of a flat F, $F \subseteq Q$:*

(i) $\dfrac{q^n-1}{q-1}$, $\dfrac{n-2}{2}$, *parabolic*

(ii) $\dfrac{(q^{(n+1)/2}-1)(q^{(n+1)/2}+1)}{q-1}$, $\dfrac{n-1}{2}$ *hyperbolic,*

(iii) $\dfrac{(q^{(n+1)/2}-1)(q^{(n+1)/2}+1)}{q-1}$, $\dfrac{n-3}{2}$ *elliptic.*

Classical generalized quadrangles

due to J. Tits (all associated with classical groups)

An **orthogonal** generalized quadrangle $\boldsymbol{Q(d,q)}$ is determined by isotropic points and lines of a nondegenerate quadratic form in

$$\mathrm{PG}(d,q), \quad \text{for } d \in \{3,4,5\}.$$

For $d = 3$ we have $t = 1$. An orthogonal generalized quadrangle $Q(4, q)$ has parameters $\boldsymbol{(q, q)}$.

Its dual is called **symplectic** (or **null**)
generalized quadrangle $\boldsymbol{W(q)}$
(since it can be defined on points of PG$(3, q)$, together with the self-polar lines of a null polarity),

and it is for even q isomorphic to $Q(4, q)$.

Let H be a nondegenerate hermitian variety (e.g., $V(x_0^{q+1} + \cdots + x_d^{q+1})$) in $\mathrm{PG}(d, q^2)$.

Then its points and lines form a generalized quadrangle called a **unitary** (or **Hermitean**) generalized quadrangle $\boldsymbol{\mathcal{U}(d, q^2)}$.

A unitary generalized quadrangle $\mathcal{U}(3, q^2)$ has parameters $\boldsymbol{(q^2, q)}$ and is isomorphic to a dual of orthogonal generalized quadrangle $Q(5, q)$.

Finally, we describe one more construction (Ahrens, Szekeres and independently M. Hall)

Let $\mathcal{O}$ be a hyperoval of $\mathrm{PG}(2,q)$, $q=2^h$, i.e., (i.e., a set of $q{+}2$ points meeting $\forall$ line in 0 or 2 points), and imbed $\mathrm{PG}(2,q)\!=\!H$ as a plane in $\mathrm{PG}(3,q)\!=\!P$.

Define a generalized quadrangle $\boldsymbol{T_2^*(\mathcal{O})}$ with parameters

$$\boldsymbol{(q-1,q+1)}$$

by taking for points just the points of $P\backslash H$, and for lines just the lines of P which are not contained in H and meet $\mathcal{O}$ (necessarily in a unique point).

For a systematic combinatorial treatment of generalized quadrangles we recommend the book by **Payne and Thas**.

The order of each known generalized quadrangle or its dual is one of the following: $(s,1)$ for $s \geq 1$;

$$\begin{gathered}(q,q),\\(q,q^2),\\(q^2,q^3),\\(q-1,q+1),\end{gathered}$$

where q is a prime power.

Small examples

$s = 2$: !(2,2), !(2,4)

$s = 3$: $(3,3) = W(3)$ or $Q(4,3)$,
$(3,5) = T_2^*(\mathcal{O})$,
$(3,9) = Q(5,3)$

$s = 4$: $(4,4) = W(4)$,

one known example for each (4,6), (4,8), (4,16)

existence open for (4,11), (4,12).

The flag geometry of a generalized polygon $\mathcal{G}$ has as pts the vertices of $\mathcal{G}$ (of both types), and as lines the flags of $\mathcal{G}$, with the obvious incidence.

It is easily checked to be a generalised $2n$-gon in which every line has two points; and any generalised $2n$-gon with two points per line is the flag geometry of a generalised n-gon.

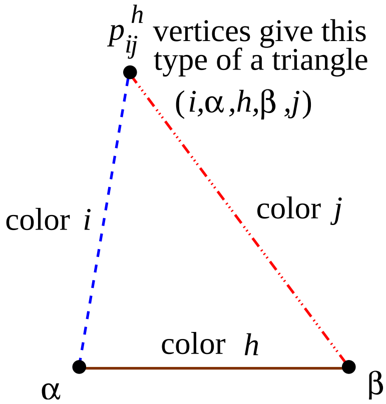
Theorem (Feit and Higman). *A thick generalised n-gon can $\exists$ only for $n = 2, 3, 4, 6$ or 8.*

Additional information:

- *if $n = 4$ or $n = 8$, then $t \leq s^2$ and $s \leq t^2$;*
- *if $n = 6$, then st is a square and $t \leq s^3$, $s \leq t^3$.*
- *if $n = 8$, then $2st$ is a square.*

V. Association schemes

- definition
- Bose-Mesner algebra
- examples
- symmetry
- duality
- Krein conditions



colored triangles over a fixed base

For a pair of given ***d*-tuples** $\boldsymbol{a}$ in $\boldsymbol{b}$ over an alphabet with $n \geq 2$ symbols, there are $d+1$ possible relations:

they can be *equal*, they can *coincide* on $d-1$ places, $d-2$ places, ..., or they can be *distinct* on all the places.

For a pair of given ***d*-subsets** A and B of the set with n elements, where $n \geq 2d$, we have $d+1$ possible relations:

they can be *equal*, they can *intersect* in $d-1$ elements, $d-2$ elements, ..., or they can be *disjoint*.

The above examples, together with the list of relations are examples of **association schemes** that we will introduce shortly.

In 1938 **Bose** and **Nair** introduced association schemes for applications in statistics.

However, it was **Philippe Delsarte** who showed in his thesis that association schemes can serve as a common framework for problems ranging from error-correcting codes, to combinatorial designs. Further connections include

– group theory (primitivity and imprimitivity),
– linear algebra (spectral theory),
– metric spaces,
– study of duality
– character theory,
– representation and orthogonal polynomials.

Bannai and Ito:

We can describe algebraic combinatorics as

"a study of combinatorial objects
with theory of characters"

or as

"a study of groups without a group" .

Even more connections:

– knot theory (spin modules),
– linear programming bound,
– finite geometries.

A (symmetric) ***d*-class association scheme** on n vertices is a set of binary symmetric $(n \times n)$-matrices $I = A_0, A_1, \dots, A_d$ s.t.

(a) $\sum_{i=0}^{d} A_i = J$, where J is the all-one matrix,

(b) for all $i, j \in \{0, 1, \dots, d\}$ the product $A_i A_j$ is a linear combination of the matrices $A_0, \dots, A_d$.

It is essentially a colouring of the edges of the complete graph K_n with d colours, such that the number of triangles with a given colouring on a given edge depends only on the colouring and not on the edge.

Bose-Mesner algebra

Subspace of $n \times n$ dim. matrices over $\mathbb{R}$ generated by $A_0, \ldots, A_d$ is, by (b), a *commutative algebra*, known as the **Bose-Mesner algebra** of $\mathcal{A}$ and denoted by $\boldsymbol{\mathcal{M}}$.

Since A_i is a symmetric binary matrix, it is the adjacency matrix of an (undirected) graph Γ_i on n vertices.

If the vertices x and y are connected in Γ_i, we will write $\boldsymbol{x\,\Gamma_i\,y}$ and say that they are in ***i*-th relation**.

The condition (a) implies that for every vertices x and y there exists a unique i, such that $x\,\Gamma_i\,y$, and that Γ_i, $i \neq 0$, has no loops.

The condition (b) implies that there exist such constants p_{ij}^h , $i, j, h \in \{0, \dots, d\}$, that

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h. \tag{1}$$

They are called **intersection numbers** of the association scheme $\mathcal{A}$. Since matrices A_i are symmetric, they commute. Thus also $p_{ij}^h = p_{ji}^h$.

By (1), the combinatorial meaning of intersection numbers p^h_{ij}, implies that they are integral and nonnegative.

Suppose $x\,\Gamma_h\,y$. Then

$$p^h_{ij} = |\{z\,;\; z\,\Gamma_i\,x \;\text{ in }\; z\,\Gamma_j\,y\}|. \tag{2}$$

Therefore, Γ_i is regular graph of valency $k_i := p^0_{ii}$ and we have $p^0_{ij} = \delta_{ij} k_i$.

By counting in two different ways all triples (x, y, z), such that

$$x\,\Gamma_h\,y, \qquad z\,\Gamma_i\,x \quad \text{and} \quad z\,\Gamma_j\,y$$

we obtain also $k_h\, p^h_{ij} = k_j\, p^j_{ih}$.

Examples

Let us now consider some examples of associative schemes.

A scheme with one class consists of the identity matrix and the adjacency matrix of a graph, in which every two vertices are adjacent, i.e., a graph of diameter 1, i.e., the completer graph K_n.

We will call this scheme **trivial**.

Hamming scheme $H(d, n)$

Let $d, n \in \mathbb{N}$ and $\Sigma = \{0, 1, \ldots, n-1\}$.

The vertex set of the association scheme $H(d, n)$ are all d-tuples of elements on Σ. Assume $0 \le i \le d$.

Vertices x and y are in i-th relation iff they differ in i places.

We obtain a d-class association scheme on n^d vertices.

Bilinear Forms Scheme $\mathcal{M}_{d\times m}(q)$

(a variation from linear algebra) Let $d, m \in \mathbb{N}$ and q a power of some prime.

All $(d \times m)$-dim. matrices over $\mathrm{GF}(q)$ are the vertices of the scheme,

two being in i-th relation, $0 \leq i \leq d$,
when the rank of their difference is i.

Johnson Scheme $J(n, d)$

Let $d, n \in \mathbb{N}$, $d \leq n$ and X a set with n elements.

The vertex set of the association scheme $J(n, d)$ are all d-subsets of X.

Vertices x and y are in i-th $0 \leq i \leq \min\{d, n-d\}$, relation iff their intersection has $d - i$ elements.

We obtain an association scheme with $\min\{d, n-d\}$ classes and on $\binom{n}{d}$ vertices.

q-analog of Johnson scheme $J_q(n, d)$ (Grassman scheme)

The vertex set consists of all d-dim.
subspaces of n-dim. vector space V over GF(q).

Two subspaces A and B of dim. d are in i-th relation, $0 \le i \le d$, when $\dim(A \cap B) = d - i$.

Cyclomatic scheme

Let q be a prime power and d a divisor of $q-1$.

Let C_1 be a subgroup of the multiplicative group of the finite field $\mathrm{GF}(q)$ with index d, and let $C_1, \dots, C_d$ be the cosets of the subgroup C_1.

The vertex set of the scheme are all elements of $\mathrm{GF}(q)$, x and y being in i-th relation when $x-y \in C_i$ (and in 0 relation when $x = y$).

We need $-1 \in C_1$ in order to have symmetric relations, so $2 \mid d$, if q is odd.

How can we verify if some set of matrices represents an association scheme?

The condition (b) does not need to be verified directly.

It suffices to check that the RHS of (2) is independent of the vertices (without computing p_{ij}^h).

We can use *symmetry*.

Let X be the vertex set and $\Gamma_1, \dots, \Gamma_d$ the set of graphs with $V(\Gamma_i) = X$ and whose adjacency matrices together with the identity matrix satisfies the condition (a).

$\boxed{\text{Symmetry}}$

An **automorphism** of this set of graphs is a permutation of vertices, that preserves adjacency.

Adjacency matrices of the graphs $\Gamma_1, \ldots, \Gamma_d$, together with the identity matrix is an association scheme if

> $\forall i$ the automorphism group acts transitively on pairs of vertices that are adjacent in Γ_i

(this is a sufficient condition).

Primitivity and imprimitivity

A d-class association scheme is **primitive**, if all its graphs Γ_i, $1 \leq i \leq d$, are connected, and **imprimitive** othervise.

The trivial scheme is primitive.

Johnson scheme $J(n, d)$ is primitive iff $n \neq 2d$.
In the case $n = 2d$ the graph Γ_d is disconnected.

Hamming scheme $H(d, n)$ is primitive iff $n \neq 2$.
In the case $n = 2$ the graphs Γ_i, $1 \leq i \leq \lfloor d/2 \rfloor$, and the graph Γ_d are disconnected.

Let $\{A_0, \ldots, A_d\}$ be a d-class associative scheme $\mathcal{A}$ and let π be a partition of $\{1, \ldots, d\}$ on $m \in \mathbb{N}$ nonempty cells. Let $A'_1, \ldots, A'_m$ be the matrices of the form

$$\sum_{i \in C} A_i,$$

where C runs over all cells of partition π, and set $A'_0 = I$. These binary matrices are the elements of the Bose-Mesner algebra $\mathcal{M}$, they commute, and their sum is J.

Very often the form an associative scheme, denoted by $\mathcal{A}'$, in which case we say that $\mathcal{A}'$ was obtained from $\mathcal{A}$ by **merging** of classes (also by **fusion**).

For $m = 1$ we obtain the trivial associative scheme.

Brouwer and Van Lint used merging to construct some new 2-class associative schemes (i.e., $m = 2$).

For example, in the Johnson scheme $J(7,3)$ we merge A_1 and A_3 to obtain a strongly regular graph, which is the line graph of PG$(3,2)$.

Two bases and duality

Theorem. *Let $\mathcal{A} = \{A_0, \dots, A_d\}$ be an associative scheme on n vertices. Then there exists orthogonal idempotent matrices $E_0, \dots, E_d$ and $p_i(j)$, such that*

(a) $$\sum_{j=0}^{d} E_j = I,$$

(b) $$A_i E_j = p_i(j) E_j, \quad \text{i.e.,} \quad A_i = \sum_{j=0}^{d} p_i(j) E_j,$$

(c) $$E_0 = \frac{1}{n} J,$$

(d) *matrices $E_0, \dots, E_d$ are a basis of a $(d+1)$-dim. vector space, generated by $A_0, \dots, A_d$.*

Proof. Let $i \in \{1, 2, \ldots, d\}$. From the spectral analysis of normal matrices we know that $\forall A_i$ there exist pairwise orthogonal idempotent matrices Y_{ij} and real numbers θ_{ij}, such that $A_i Y_{ij} = \theta_{ij} Y_{ij}$ and

$$\sum_j Y_{ij} = I. \tag{3}$$

Furthermore, each matrix Y_{ij} can be expressed as a polynomial of the matrix A_i.

Since $\mathcal{M}$ is a commutative algebra, the matrices Y_{ij} commute and also commute with matrices $A_0, \ldots, A_d$.

Therefore, each product of this matrices is an idempotent matrix (that can be also 0).

We multiply equations (3) for $i = 1, \ldots, d.$ to obtain an equation of the following form

$$I = \sum_j E_j, \tag{4}$$

where each E_j is an idempotent that is equal to a product of d idempotents Y_{ik_i}, where Y_{ik_i} is an idempotent from the spectral decompozition of A_i.

Hence, the idempotents E_j are pairwise orthogonal, and for each matrix A_i there exists $p_i(j) \in \mathbb{R}$, such that $A_i E_j = p_i(j) E_j$.

Therefore,

$$A_i = A_i I = A_i \sum_j E_j = \sum_j p_i(j) E_j.$$

This tells us that each matrix A_i is a linear combination of the matrices E_j.

Since the nonzero matrices E_j are pairwise othogonal, they are also linearly independent.

Thus they form a basis of the BM-algebra $\mathcal{M}$, and there is exactly $d+1$ nonzero matrices among E_j's.

The proof of (c) is left for homework. $\blacksquare$

The matrices $E_0, \dots, E_d$ are called **minimal idempotents** of the associative scheme $\mathcal{A}$. **Schur** (or Hadamard) product of matrices is an entry-wise product. denoted by "$\circ$". Since $A_i \circ A_j = \delta_{ij} A_i$, the BM-algebra is closed for Schur product.

The matrices A_i are pairwise othogonal idempotents for Schure multiplication, so they are also called **Schur idempotents** of $\mathcal{A}$.

Since the matrices $E_0, \dots, E_d$ are a basis of the vector space spanned by $A_0, \dots, A_d$, also the following statement follows.

Corollary. *Let $\mathcal{A} = \{A_0, \dots, A_d\}$ be an associative scheme and $E_0, \dots, E_d$ its minimal idempotents. Then $\exists\ q_{ij}^h \in \mathbb{R}$ and $q_i(h) \in \mathbb{R}$ $(i, j, h \in \{0, \dots, d\})$, such that*

(a) $$E_i \circ E_j = \frac{1}{n} \sum_{h=0}^{d} q_{ij}^h E_h,$$

(b) $$E_i \circ A_j = \frac{1}{n} q_i(j) A_j, \quad \text{i.e.,} \quad E_i = \frac{1}{n} \sum_{h=0}^{d} q_i(h) A_h,$$

(c) *matrices A_i have at most $d+1$ distinct eigenvalues.* ■

There exists a basis of $d+1$ (orthogonal) minimal idempotents E_i of the BM-algebra $\mathcal{M}$ such that

$$E_0 = \frac{1}{n}J \quad \text{and} \quad \sum_{i=0}^{d} E_i = I,$$

$$E_i \circ E_j \; = \; \frac{1}{n}\sum_{h=0}^{d} q_{ij}^h E_h, \quad A_i \; = \; \sum_{h=0}^{d} p_i(h) E_h$$

$$\text{and} \quad E_i \; = \; \frac{1}{n}\sum_{h=0}^{d} q_i(h) A_h \;\; (0 \le i, j \le d),$$

The parameters q_{ij}^h are called **Krein parameters**, $p_i(0), \dots, p_i(d)$ are **eigenvalues** of matrix A_i, and $q_i(0), \dots, q_i(d)$ are the **dualne eigenvalues** of E_i.

The **eigenmatrices** of the associative scheme $\mathcal{A}$ are $(d+1)$-dimensional square matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ defined by

$$(P)_{ij} = p_j(i) \quad \text{and} \quad (Q)_{ij} = q_j(i).$$

By setting $j = 0$ in the left identity of Corollary (b) and taking traces, we see that the eigenvalue $p_i(1)$ of the matrix A_1 has multiplicity $m_i = q_i(0) = \text{rank}(E_i)$.

By Theorem (b) and Corollary (b), we obtain

$$\boldsymbol{PQ = nI = QP}.$$

There is another relation between P and Q.

Take the trace of the identity in Theorem (b):

$$\boldsymbol{\Delta_k Q = P^T \Delta_m},$$

where Δ_k and Δ_m are the diagonal matrices with entries $(\Delta_k)_{ii} = k_i$ and $(\Delta_m)_{ii} = m_i$.

This relation implies $P\Delta_k^{-1}P^T = n\Delta_m^{-1}$, and by comparing the diagonal entries also

$$\sum_{h=0}^{d} p_h(i)^2/k_h = n/m_i.$$

which gives us an expression for the multiplicity m_i in terms of eigenvalues.

Using the eigenvalues we can express all intersection numbers and Krein parameters.

For example, if we multiply the equality in Corollary (a) by E_h, we obtain

$$q_{ij}^h \, E_h = nE_h(E_i \circ E_j),$$

i.e.,

$$\begin{aligned} q_{ij}^h &= \frac{n}{m_h} \operatorname{trace}(E_h(E_i \circ E_j)) \qquad (5) \\ &= \frac{n}{m_h} \operatorname{sum}(E_h \circ E_i \circ E_j), \qquad (6) \end{aligned}$$

where the sum of a matrix is equal to the sum of all of its elements.

By Corollary (b), it follows also

$$E_i \circ E_j \circ E_h = \frac{1}{n^3} \sum_{\ell=0}^{d} q_i(\ell)\, q_j(\ell)\, q_h(\ell)\, A_\ell,$$

therefore, by $\Delta_k Q = (\Delta_m P)^T$, we obtain

$$\begin{aligned} q_{ij}^h \;&=\; \frac{1}{nm_h} \sum_{\ell=0}^{d} q_i(\ell)\, q_j(\ell)\, q_h(\ell)\, k_\ell \\ &=\; \frac{m_i m_j}{n} \sum_{\ell=0}^{d} \frac{p_\ell(i)\, p_\ell(j)\, p_\ell(h)}{k_{\ell^2}.} \end{aligned}$$

Krein parameters satisfy the so-called
Krein conditions:

Theorem [Scott].
Let $\mathcal{A}$ be an associative scheme with n vertices and $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ the standard basis in $\mathbb{R}^n$. Then

$$\boxed{q_{ij}^h \geq 0.}$$

Moreover, for $\boldsymbol{v} = \sum_{i=1}^{n} \boldsymbol{e}_i \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_i$, we have

$$q_{ij}^h = \frac{n}{m_h} \|(E_i \otimes E_j \otimes E_h)\boldsymbol{v}\|^2,$$

and $q_{ij}^h = 0$ iff $(E_i \otimes E_j \otimes E_h)\boldsymbol{v} = 0$.

PROOF (Godsil's sketch). Since the matrices E_i are pairwise orthogonal idempotents, we derive from Corollary (b) (by multiplying by E_h)

$$(E_i \circ E_j)E_h \;=\; \frac{1}{n}\, q_{ij}^h E_h.$$

Thus q_{ij}^h/n is an eigenvalue of the matrix $E_i \circ E_j$ on a subspace of vectors that are determined by the columns of E_h.

The matrices E_i are positive semidefinite (since they are symmetric, and all their eigenvalues are 0 or 1).

On the other hand, the Schur product of semidefinite matrices is again semidefinite, so the matrix $E_i \circ E_j$ has nonnegative eigenvalues. Hence, $q_{ij}^h \geq 0$.

By (5) and the well known tensor product identity

$$(A \otimes B)(x \otimes y) = Ax \otimes By$$

for $A, B \in \mathbb{R}^{n\times n}$ and $x, y \in \mathbb{R}^n$, we obtain

$$\begin{aligned} q_{ij}^h &= \frac{n}{m_h}\,\mathrm{sum}(E_i \circ E_j \circ E_h) \\ &= \frac{n}{m_h}\,\boldsymbol{v}^T(E_i \otimes E_j \otimes E_h)\boldsymbol{v}. \end{aligned}$$

Now the statement follows from the fact that

$$E_i \otimes E_j \otimes E_h$$

is a symmetric idempotent. ■

Another strong criterion for an existence of associative schemes is an **absolute bound**, that bounds the rank of the matrix $E_i \circ E_j$.

Theorem. *Let $\mathcal{A}$ be a d-class associative scheme. Then its multiplicities m_i, $1 \le i \le d$, satisfy inequalities*

$$\sum_{q_{ij}^h \neq 0} m_h \le \begin{cases} m_i m_j & \text{if } i \neq j, \\ \frac{1}{2} m_i (m_i + 1) & \text{if } i = j. \end{cases}$$

PROOF (sketch). The LHS is equal to the $\operatorname{rank}(E_i \circ E_j)$ and is greater or equal to the

$$\operatorname{rank}(E_i \otimes E_j) = m_i m_j.$$

Suppose now $i = j$. Among the rows of the matrix E_i we can choose m_i rows that generate all the rows.

Then the rows of the matrix $E_i \circ E_i$, whose elements are the squares of the elements of the matrix E_i, are generated by

$$m_i + \binom{m_i}{2} \text{ rows,}$$

that are the Schur products of all the pairs of rows among all the m_i rows. ■

An association scheme $\mathcal{A}$ is ***P*-polynomial** (called also **metric**) when there there exists a permutation of indices of A_i's, s.t.

$$\exists \text{ polynomials } p_i \text{ of degree } i \text{ s.t. } A_i = p_i(A_1),$$

i.e., the intersection numbers satisfy the **Δ-condition**

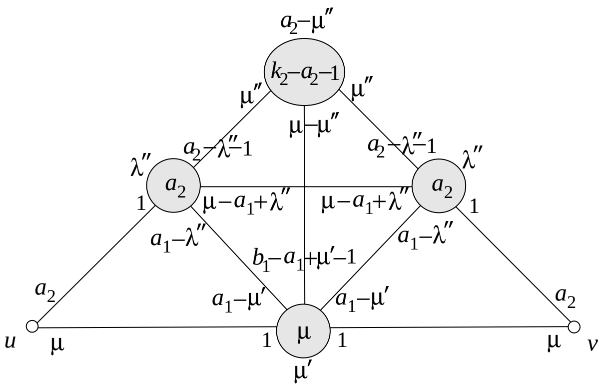
(that is, $\forall\ i, j, h \in \{0, \ldots, d\}$

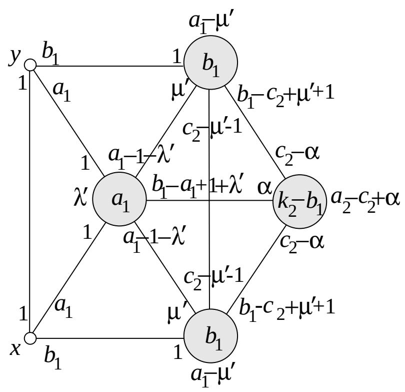
- $p_{ij}^h \neq 0$ implies $h \leq i + j$ and
- $p_{ij}^{i+j} \neq 0$).

An associative scheme $\mathcal{A}$ is ***Q*-polynomial** (called also **cometric**) when there exists a permutation of indices of E_i's, s.t. the Krein parameters q_{ij}^h satisfy the Δ-condition.

Theorem. **[Cameron, Goethals and Seidel]**

In a strongly regular graph vanishing of either of Krein parameters q_{11}^1 and q_{22}^2 implies that first and second subconstituent graphs are strongly regular.





SRG$(162, 56, 10, 24)$, denoted by Γ,
is unique by Cameron, Goethals and Seidel.

vertices: special vertex ∞,
56 hyperovals in PG$(2, 4)$ in a $L_3(4)$-orbit,
105 flags of PG$(2, 4)$

adjacency: ∞ is adjacent to the hyperovals

$$\begin{array}{rcl} \text{hyperovals } \mathcal{O} \sim \mathcal{O}' & \iff & \mathcal{O} \cap \mathcal{O}' = \emptyset \\ (p, L) \sim \mathcal{O} & \iff & |\mathcal{O} \cap L \backslash \{p\}| = 2 \\ (p, L) \sim (q, M) & \iff & p \neq q,\ L \neq M \text{ and} \\ & & (p \in M \text{ or } q \in L). \end{array}$$

The hyperovals induce the Gewirtz graph,
i.e., the unique SRG(56,10,0,2))
and the flags induce a SRG(105,32,4,12).

VI. Equitable partitions

- definition
- quotients
- eigenvectors
- (antipodal) covers

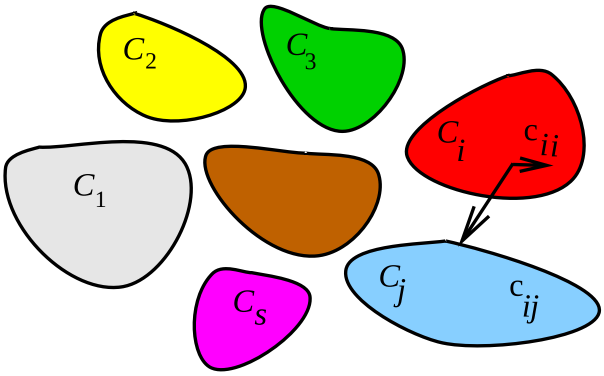
$$\begin{vmatrix} 1 & a & b & c & d & e \\ a & 1 & a & b & c & d \\ b & a & 1 & a & b & c \\ c & b & a & 1 & a & b \\ d & c & b & a & 1 & a \\ e & d & c & b & a & 1 \end{vmatrix}$$

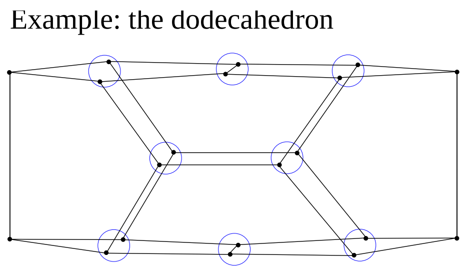
$$= \begin{vmatrix} 1+a & a+b & b+c \\ a+b & 1+c & a+d \\ b+c & a+d & 1+e \end{vmatrix} \begin{vmatrix} 1-a & a-b & b-c \\ a-b & 1-c & a-d \\ b-c & a-d & 1-e \end{vmatrix}$$

An **equitable partition** of a graph Γ is a partition of the vertex set $V(\Gamma)$ into **parts** $C_1, C_2, \ldots, C_s$ s.t.

(a) vertices of each part C_i induce a *regular* graph,

(b) edges between C_i and C_j induce a *half-regular* graph.



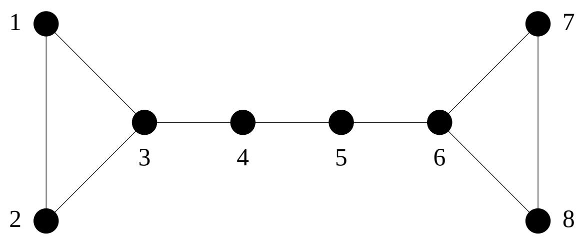


Numbers c_{ij} are the parameters of the partition.

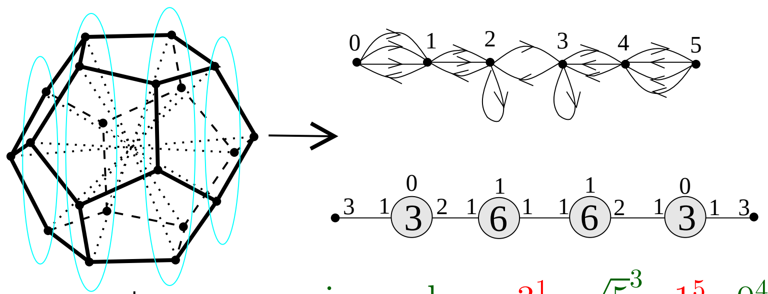
Orbits of a group acting on Γ form an equitable partition.

But not all equitable partitions come from groups:

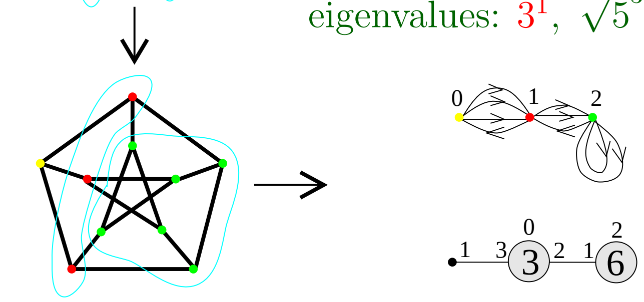
$$\{\{1,2,4,5,7,8\},\{3,6\}\}.$$



Equitable partitions give rise to **quotient** graphs $\boldsymbol{G/\pi}$, which are directed multigraphs with cells as vertices and c_{ij} arcs going from C_i to C_j.



eigenvalues: 3^1, $\sqrt{5}^3$, 1^5, 0^4, -2^4, $-\sqrt{5}^3$



Set $\boldsymbol{X} := V\Gamma$ and $\boldsymbol{n} := |X|$. Let $\boldsymbol{V} = \mathbb{R}^n$ be the vector space over $\mathbb{R}$ consisting of all column vectors whose coordinates are indexed by X.

For a subset $S \subseteq X$ let its **characteristic vector** be an element of V, whose coordinates are equal 1 if they correspond to the elements of S and 0 otherwise.

Let $\pi = \{C_1, \dots, C_s\}$ be a partition of X.
The **characteristic matrix** P of π is $(n \times s)$ matrix, whose column vectors are the characteristic vectors of the parts of π (i.e., $P_{ij} = 1$ if $i \in C_j$ and 0 otherwise).

Let **$\mathbf{Mat}_{\boldsymbol{X}}(\mathbb{R})$** be the $\mathbb{R}$-algebra consisting of all real matrices, whose rows and columns are indexed by X. Let $\boldsymbol{A} \in \mathrm{Mat}_X(\mathbb{R})$ be the adjacency matrix of Γ.

$\mathrm{Mat}_X(\mathbb{R})$ acts on V by left multiplication.

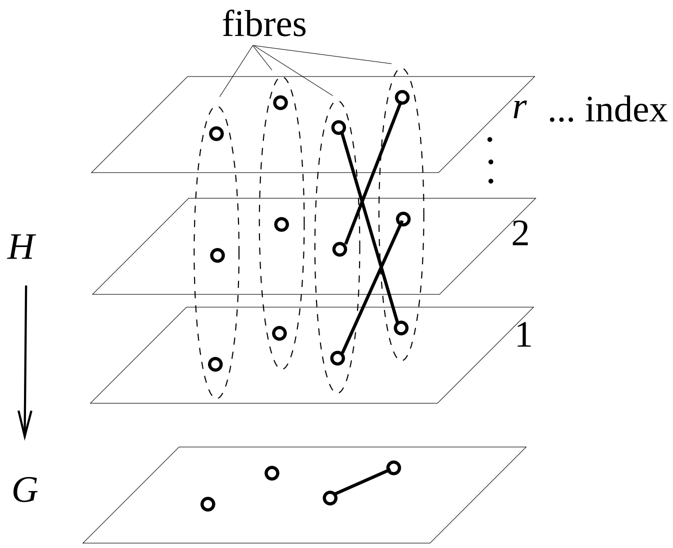
Theorem. *Let π be a partition of $V\Gamma$ with the characteristic matrix P. TFAE*
(i) π equitable,
(ii) $\exists$ a $s \times s$ matrix B s.t. $A(\Gamma)P = PB$
(iii) the span(col(P)) is $A(\Gamma)$-invariant.
If π is equitable then $B = A(\Gamma/\pi)$.

Theorem. *Assume $AP = PB$.*
(a) If $B\boldsymbol{x} = \theta\boldsymbol{x}$, then $AP\boldsymbol{x} = \theta P\boldsymbol{x}$.
(b) If $A\boldsymbol{y} = \theta\boldsymbol{y}$, then $\boldsymbol{y}^T PB = \theta\boldsymbol{y}^T P$.
(c) The characteristic polynomial of matrix B divides the characteristic polynomial of matrix A.

An eigenvector x of Γ/π corresponding to θ extends to an eigenvector of Γ, which is constant on parts, so

$$m_\theta(\Gamma/\pi) \le m_\theta(\Gamma).$$

$\tau \in \mathrm{ev}(\Gamma)\backslash\mathrm{ev}(\Gamma/\pi) \implies$ each eigenvector of Γ corresponding to τ sums to zero on each part.

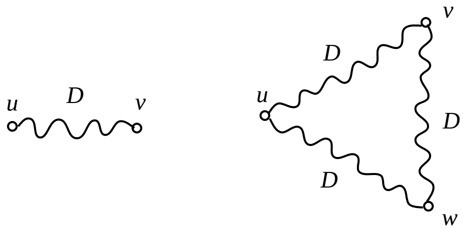


H is an ***r*-cover** if there is a partition of $V(H)$ into independent sets, called **fibres**, such that there is either a matching or nothing between any two fibres.

$\Gamma = H/\pi$ ... **quotient** (corresponding to π)

H graph, $\quad D$ diameter

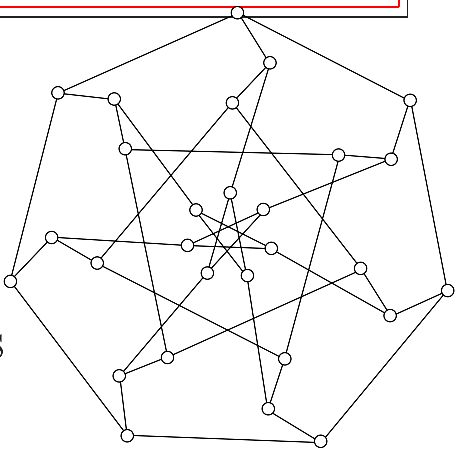
If being at distance 0 or D is an equivalence relation on $V(H)$, we say that H is **antipodal**.



If an antipodal graph H covers H/π and π consists of antipodal classes, then H is called **antipodal cover**.

VII. Distance-regular graphs

- distance-regularity
- intersection numbers
- eigenvalues and cosine sequences
- classification
- classical infinite families
- antipodal distance-regular graphs



COXETER GRAPH

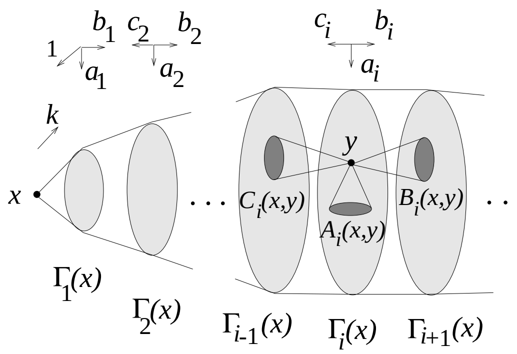
unique, cubic, DT

28 vertices, diameter 4, girth 7

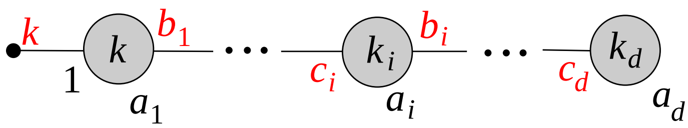
Aut=PGL(2,7), pt stab. D_{12}

Distance-regularity:

Γ graph, diameter d, $\forall\, x \in V(\Gamma)$
the **distance partition** $\{\Gamma_0(x), \Gamma_1(x), \ldots, \Gamma_d(x)\}$ corresponding to x

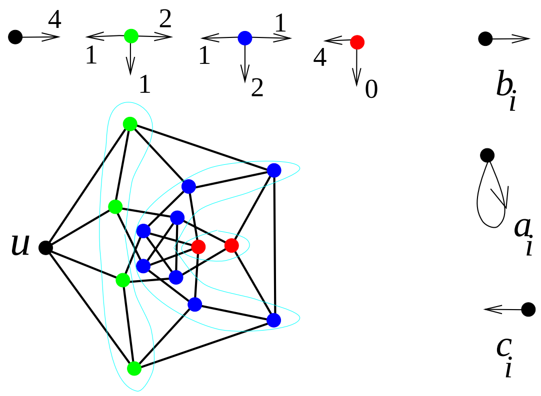


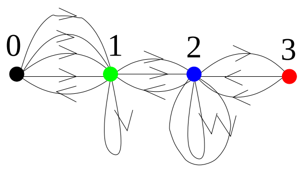
is *equitable* and the **intersection array** $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is *independent* of x.



A small example of a distance-regular graph:

(antipodal, i.e., being at distance diam. is a transitive relation)

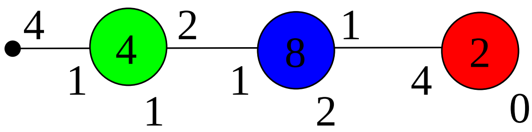




$$\begin{bmatrix} 0 & b_0 & & \\ c_1 & a_1 & b_1 & \\ & c_2 & a_2 & b_2 \\ & & c_3 & a_3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & & \\ 1 & 1 & 2 & \\ & 1 & 2 & 1 \\ & & 4 & 0 \end{bmatrix}$$

The above parameters are the same for each vertex u:

$\{b_0, b_1, b_2; c_1, c_2, , c_3\}$
$= \{4, 2, 1; 1, 1, 4\}.$



Intersection numbers

Set $\boldsymbol{p_{ij}^h} := |\Gamma_i(u) \cap \Gamma_j(v)|$, where $\partial(u,v) = h$. Then

$$a_i = p_{i1}^i, \qquad b_i = p_{i+1,1}^i, \qquad c_i = p_{i-1,1}^i, \qquad k_i = p_{ii}^0,$$

$k_i = p_{i0}^h + \cdots + p_{id}^h$ and in particular $a_i + b_i + c_i = k$.

A connected graph is **distance-transitive** when any pair of its vertices can be mapped (by a graph authomorphism, i.e., an adjacency preserving map) to any other pair of its vertices at the same distance.

distance-transitivity $\Longrightarrow$ distance-regularity

All the intersection numbers are determined by the numbers in the **intersection array**

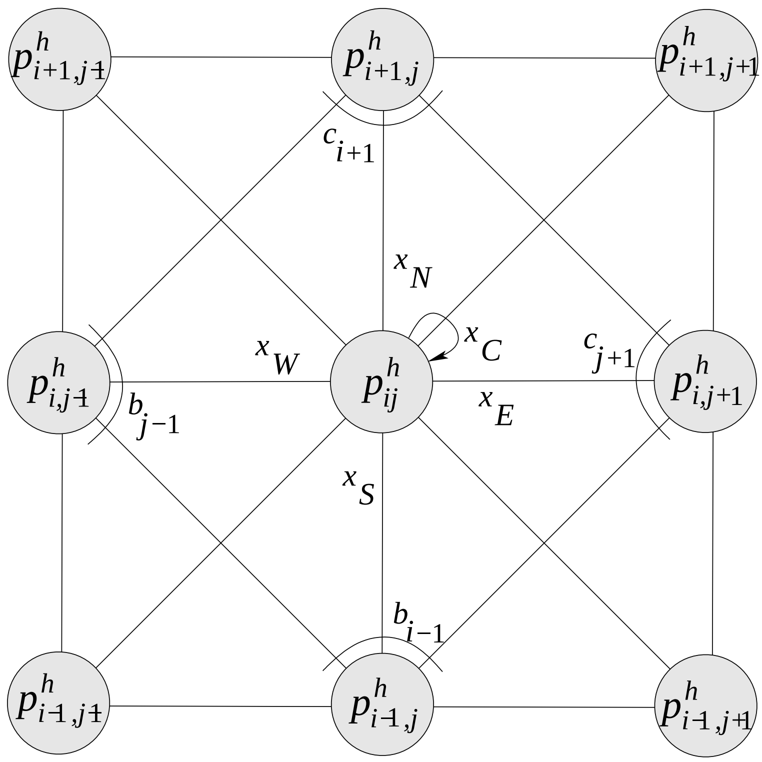
$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$$

of Γ. This can be proved by induction on i, using the following recurrence relation:

$$c_{j+1}p^h_{i,j+1} + a_j p^h_{ij} + b_{j-1}p^h_{i,j-1} = c_{i+1}p^h_{i+1,j} + a_i p^h_{ij} + b_{i-1}p^h_{i-1,j}$$

obtained by a 2-way counting for vertices u and v at distance h of edges with one end in $\Gamma_i(u)$ and another in $\Gamma_j(v)$ (see the next slide). Therefore,

the intersection numbers do not depend on the choice of vertices u and v at distance r.



$$x_W + x_C + x_E = a_i \quad \text{and} \quad x_N + x_C + x_S = a_j.$$

An arbitrary list of numbers b_i and c_i does not determine a distance-regular graph.

It has to satisfy numerous feasiblity conditions (e.g. all intersection numbers have to be integral).

One of the main questions of the theory of distance-regular graphs is for a given intersection array

- **to construct a distance-regular graph,**
- **to prove its uniqueness,**
- **to prove its nonexistence.**

Some basic properties of the intersection numbers will be collected in the following result.

Lemma. Γ *distance-regular, diameter* d *and intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$. *Then*

(i) $b_0 > b_1 \geq b_2 \geq \cdots \geq b_{d-1} \geq 1$,

(ii) $1 = c_1 \leq c_2 \leq \cdots \leq c_d$,

(iii) $b_{i-1}k_{i-1} = c_i k_i$ *for* $1 \leq i \leq d$,

(iv) *if* $i + j \leq d$, *then* $c_i \leq b_j$,

(v) *the sequence* $k_0, k_1, \ldots, k_d$ *is unimodal, (i.e., there exists such indices* h, ℓ $(1 \leq h \leq \ell \leq d)$, *that* $k_0 < \cdots < k_h = \cdots = k_\ell > \cdots > k_d$.

PROOF. (i) Obviously $b_0 > b_1$. Set $2 \le i \le d$. Let $v, u \in V\Gamma$ be at distance d and $v = v_0, v_1, \ldots, v_d = u$ be a path. The vertex v_i has b_i neighbours, that are at distance $i+1$ from v. All these b_i vertices are at distance i from v_1, so $b_{i-1} \ge b_i$.

(iii) The number of edges from $\Gamma_{i-1}(v)$ to $\Gamma_i(v)$ is $b_{i-1}k_{i-1}$, while from $\Gamma_i(v)$ to $\Gamma_{i-1}(v)$ is c_ik_i.

(iv) The vertex v_i has c_i neighbours, that are at distance $i-1$ from v. All these vertices are at distance $j+1$ from v_{i+j}. Hence $c_i \le b_j$.

The statement (ii) can be proven the same way as (i), and (v) follows directly from (i), (ii) and (iii). ■

Lemma. *A connected graph G of diameter d is distance-regular iff* $\exists\ a_i$, b_i *and* c_i *such that*

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \qquad for \quad 0 \le i \le d.$$

If G is a distance-regular graph, then $A_i = v_i(A)$ for some polynomial $v_i(x)$ of degree i, for $0 \le i \le d+1$.

The sequence $\{\boldsymbol{v_i(x)}\}$ is determined with $v_{-1}(x) = 0$, $v_0(x) = 1$, $v_1(x) = x$ and for $i \in \{0, 1, \dots, d\}$ with

$$c_{i+1}v_{i+1}(x) = (x - a_i)v_i(x) - b_{i-1}v_{i-1}(x).$$

In this sense distance-regular graphs are combinatorial representation of orthogonal polynomials.

Eigenvalues

The intersection array of a distance-regular graph Γ

$$\{k, b_1, \ldots, b_{d-2}, b_{d-1}; 1, c_2, \ldots, c_{d-1}, c_d\},$$

i.e., the quotient graph Γ/π with the adjacency matrix

$$A(\Gamma/\pi) = \begin{pmatrix} a_0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ 0 & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & \cdot & \cdot & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix}, \quad \text{determines}$$

all the eigenvalues of Γ and their multiplicities.

The vector $v = \big(v_0(\theta), \dots, v_d(\theta)\big)^T$ is a left eigenvector of this matrix corresponding to the eigenvalue θ.

Similarly a vector $w = (w_0(\theta), \dots, w_d(\theta))^T$ defined by $w_{-1}(x) = 0$, $w_0(x) = 1$, $w_1(x) = x/k$ and for $i \in \{0, 1, \dots, d\}$ by

$$xw_i(x) = c_i w_{i-1}(x) + a_i w_i(x) + b_i w_{i+1}(x)$$

is a right eigenvector of this matrix, corresponding to the eigenvalue θ.

There is the following relation between coordinates of vectors w and v: $w_i(x)k_i = v_i(x)$.

For $\theta \in \mathrm{ev}(\Gamma)$ and associated primitive idempotent E:

$$E \;=\; \frac{m_\theta}{|V\Gamma|} \sum_{h=0}^{d} \omega_h A_h \quad (0 \le i \le d),$$

$\omega_0, \dots, \omega_d$ is the **cosine sequence** of E (or θ).

Lemma. *Γ distance-regular, diam. $d \ge 2$, E is a primitive idempotent of Γ corresponding to θ, $\omega_0, \dots, \omega_d$ is the cosine sequence of θ.*
For $x, y \in V\Gamma$, $i = \partial(x, y)$ we have
(i) $\quad \langle Ex, Ey \rangle =$ *xy-entry of E* $= \omega_i \dfrac{m_\theta}{|V\Gamma|}$.
(ii) $\omega_0 = 1$ *and* $\quad c_i\omega_{i-1} + a_i\omega_i + b_i\omega_{i+1} = \theta\omega_i$
for $0 \le i \le d$.

$$\omega_1 = \frac{\theta}{k}, \qquad \omega_2 = \frac{\theta^2 - a_1\theta - k}{kb_1}$$

and

$$\omega_1 - \omega_2 = \frac{(k-\theta)(a_1+1)}{kb_1}, \quad 1 - \omega_2 = \frac{(k-\theta)(\theta+b_1+1)}{kb_1}.$$

Using the Sturm's theorem for the sequence

$$b_0 \dots b_i \, \omega_i(x)$$

we obtain

Theorem. *Let $\theta_0 \geq \cdots \geq \theta_d$ be the eigenvalues of a distance regular graph. The sequence of cosines corresponding to the i-th eigenvalue θ_i has precisely i sign changes.*

Classification

Γ distance-regular, diam. d. We say Γ is **primitive**, when all the distance graphs $\Gamma_1, \dots, \Gamma_d$ are connected (and **imprimitive** otherwise).

Theorem (Smith).
An imprimitive distance-regular graph is either antipodal or bipartite.

The big project of classifying distance-regular graphs:

(a) **find all primitive distance-regular graphs,**

(b) given a distance-regular graph Γ, **find all imprimitive graphs, which give rise to Γ.**

Classical infinite families

graph	diameter	b	α	β
Johnson graph $J(n,d)$	$\min(d, n-d)$	1	1	$n-d$
Grassmann graph	$\min(k, v-k)$	q	q	$\begin{bmatrix} n-d+1 \\ 1 \end{bmatrix} - 1$
Hammin graph $H(d,n)$	d	1	0	$n-1$
Bilinear forms graph	k	q	$q-1$	q^n-1
Dual polar graph	???	q	0	q^e
Alternating forms graph	$\lfloor n/2 \rfloor$	q^2	q^2-1	q^m-1
Hermitean forms graph	n	$-r$	$-r-1$	$-(-r)^d-1$
Quadratic forms graph	$\lfloor (n+1)/2 \rfloor$	q^2	q^2-1	q^m-1

The **Gauss binomal coefficient** $\begin{bmatrix} j \\ i \end{bmatrix}$ is equal $\binom{j}{i}$ for $b = 1$ and

$$\prod_{k=0}^{i-1} \frac{b^j - b^k}{b^i - b^k}$$

otherwise.

If V is an n-dim. vector space over a finite field with b elements, then $\begin{bmatrix} n \\ m \end{bmatrix}$ is the number of m-dim. subspaces of V.

A distance-regular graph with diameter d is called **classical**, if its intersection parametres can be parametrized with four parameters (diameter d and numbers b, α and β) in the following way:

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right), \quad 0 \le i \le d-1$$

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right), \quad 1 \le i \le d,$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{j-1}.$$

Antipodal distane-regular graphs

Theorem (Van Bon and Brouwer, 1987).
Most classical distance-regular graphs have no antipodal covers.

Theorem (Terwilliger, 1993).
P- and Q-poly. association scheme with $d \geq 3$ (not C_n, $\overline{Q_n}$, $\overline{\frac{1}{2}Q_n}$ or $\overline{\frac{1}{2}J(s,2s)}$) is not the quotient of an antipodal P-polynomial scheme with $d \geq 7$.

Theorem (A.J. 1991). *H is a bipartite antipodal cover with D odd*

iff

$H \cong K_2 \otimes (H/\pi)$, *(i.e.,* **bipartite double***),*
and H/π is a generalized Odd graph.

(cf. Biggs and Gardiner, also [BCN])

A **generalized Odd graph** of diameter d is a drg, s.t. $a_1 = \cdots = a_{d-1} = 0, \quad a_d \neq 0)$

Known examples for $D = 5$ (and $d = 2$):

– Desargues graph (i.e., the Double Petersen)
– five-cube
– the Double of Hoffman-Singleton
– the Double Gewirtz
– the Double 77-graph
– the Double Higman-Sims

Theorem (Gardiner, 1974). *If H is antipodal r-cover of G, then $\iota(H)$ is (almost) determined by $\iota(G)$ and r,*

$$D_H \in \{2d_\Gamma, 2d_\Gamma + 1\} \quad and \quad 2 \le r \le k,$$

and

$$b_i = c_{D-i} \ for \ \ i = 0, \dots, D, \ i \ne d, \qquad r = 1 + \frac{b_d}{c_{D-d}}.$$

Lemma. *A distance-regular antipodal graph Γ of diameter d is a cover of its antipodal quotient with components of Γ_d as its fibres unless $d = 2$.*

Lemma. Γ *antipodal distance-regular, diameter* d. *Then a vertex* x *of* Γ, *which is at distance* $i \leq \lfloor d/2 \rfloor$ *from one vertex in an antipodal class, is at distance* $d-i$ *from all other vertices in this antipodal class. Hence*

$$\Gamma_{d-i}(x) = \cup\{\Gamma_d(y) \mid y \in \Gamma_i(x)\} \quad \text{for } 0 \leq i \leq \lfloor d/2 \rfloor .$$

For each vertex u of a cover H we denote the fibre which contains u by $F(u)$.

A **geodesic** in a graph G is a path $g_0, \ldots, g_t$, where $\mathrm{dist}(g_0, g_t) = t$.

Theorem. *G distance-regular, diameter d and parameters b_i, c_i; H its r-cover of diameter $D > 2$. Then the following statements are equivalent:*

(i) *The graph H is antipodal with its fibres as the antipodal classes (hence an antipodal cover of G) and each geodesic of length at least $\lfloor (D+1)/2 \rfloor$ in H can be extended to a geodesic of length D.*

(ii) *For any $u \in V(H)$ and $0 \le i \le \lfloor D/2 \rfloor\}$ we have*

$$S_{D-i}(u) = \cup\{F(v)\backslash\{v\} \;:\; v \in S_i(u)\}.$$

(iii) *The graph H is distance-regular with $D \in \{2d, 2d+1\}$ and intersection array*

$$\Big\{b_0, \ldots, b_{d-1}, \frac{(r-1)c_d}{r}, c_{d-1}, \ldots, c_1;$$

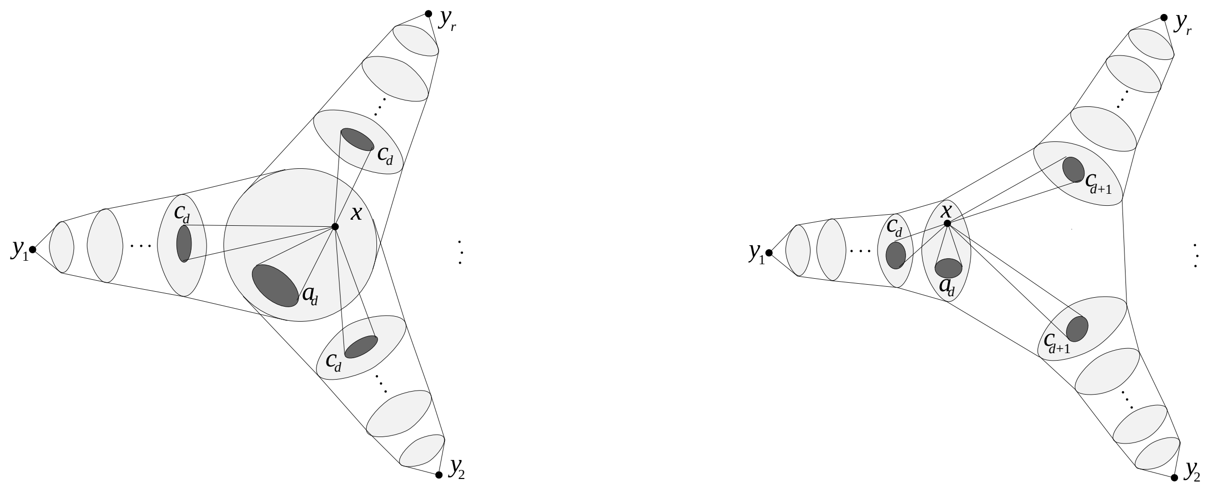
$$c_1, \ldots, c_{d-1}, \frac{c_d}{r}, b_{d-1}, \ldots, b_0\Big\} \quad \textit{for } D \textit{ even},$$

and

$$\Big\{b_0, \ldots, b_{d-1}, (r-1)t, c_d, \ldots, c_1;$$

$$c_1, \ldots, c_d, t, b_{d-1}, \ldots, b_0\Big\}$$

for D odd and some integer t.



The distance distribution corresponding to the antipodal class $\{y_1, \ldots, y_r\}$ in the case when d is even (left) and the case when d is odd (right).

Inside this partition there is a partition of the neighbourhood of the vertex x.

Theorem. *Let Γ be a distance regular graph and H a distance regular antipodal r-cover of Γ. Then every eigenvalue θ of Γ is also an eigenvalue of H with the same multiplicity.*

PROOF. Let H has diameter D, and Γ has n vertices, so $H_D = n \cdot K_r$ (K_r's corresp. to the fibres of H).

Therefore, H_D has for eigenvalues $r-1$ with multiplicity n and -1 with multiplicity $nr-n$.

The eigenvectors corresponding to eigenvalue $r-1$ are constant on fibres and those corresponding to -1 sum to zero on fibres.

Take θ to be an eigenvalue of H, which is also an eigenvalue of Γ.

An eigenvector of Γ corresponding to θ can be extended to an eigenvector of H which is constant on fibres.

We know that the eigenvectors of H are also the eigenvectors of H_D, therefore, we have $v_D(\theta) = r - 1$.

So we conclude that all the eigenvectors of H corresponding to θ are constant on fibres and therefore give rise to eigenvectors of Γ corresponding to θ. $\blacksquare$

All the eigenvalues: $A(\Gamma/\pi), N_0$ or $A(\Gamma/\pi), N_1$:

$$\begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ 0 & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & \cdot & \cdot & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix}, \begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-2} & a_{d-2} & b_{d-2} \\ & & & & c_{d-1} & a_{d-1} \end{pmatrix}$$

$$\begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ 0 & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & \cdot & \cdot & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix}, \begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d - rt \end{pmatrix}$$

Theorem. *H distance-regular antipodal r-cover, diameter D, of the distance-regular graph Γ, diameter d and parameters a_i, b_i, c_i.*

The $D-d$ eigenvalues of H which are not in $\mathrm{ev}(\Gamma)$ (the 'new' ones) are for $D = 2d$ (resp. $D = 2d+1$), the eigenvalues of the matrix N_0 (resp. N_1).

If $\theta_0 \geq \theta_1 \geq \cdots \geq \theta_D$ are the eigenvalues of H and $\xi_0 \geq \xi_1 \geq \cdots \geq \xi_d$ are the eigenvalues of Γ, then

$$\xi_0 = \theta_0, \quad \xi_1 = \theta_2, \quad \cdots, \quad \xi_d = \theta_{2d},$$

i.e., the $\mathrm{ev}(\Gamma)$ interlace the 'new' eigenvalues of H.

Connections

– **projective and affine planes**,
for $D = 3$, or $D = 4$ and $r = k$ (covers of K_n or $K_{n,n}$),

– **Two graphs** (Q-polynomial), for $D = 3$ and $r = 2$,

– **Moore graphs**, for $D = 3$ and $r = k$,

– **Hadamard matrices**, $D = 4$ and $r = 2$
(covers of $K_{n,n}$),

– **group divisible resolvable designs**,
$D = 4$ (cover of $K_{n,n}$),

– coding theory (perfect codes),
– group theory (class. of finite simple groups),
– orthogonal polynomials.

Tools:

– graph theory, counting,

– matrix theory (rank mod p),

– eigenvalue techniques,

– representation theory of graphs,

– geometry (Euclidean and finite),

– algebra and association schemes,

– topology (covers and universal objects).

Goals:

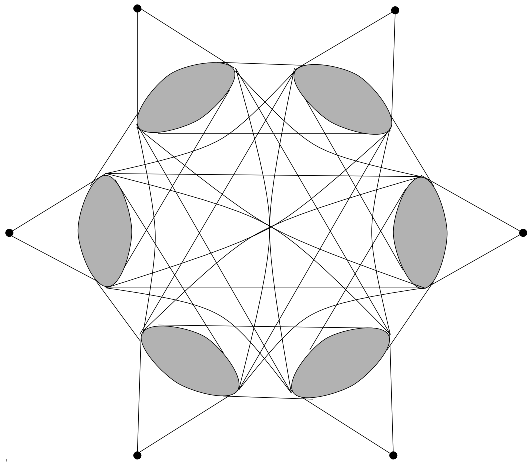
– structure of antipodal covers,
– new infinite families,
– nonexistence and uniqueness,
– characterization,
– new techniques
(which can be applied to drg or even more general)

Difficult problems:

Find a 7-cover of K_{15}.
Find a double-cover of Higman-Sims graph ($\{22, 21; 1, 6\}$).

Antipodal covers of diameter 3

Γ an antipodal distance-regular with diameter 3.
Then it is an r-cover of the complete graph K_n.
Its intersection array is $\{n-1, (r-1)c_2, 1; 1, c_2, n-1\}$.



The distance partition corresp. to an antipodal class.

Examples: 3-cube, the icosahedron.

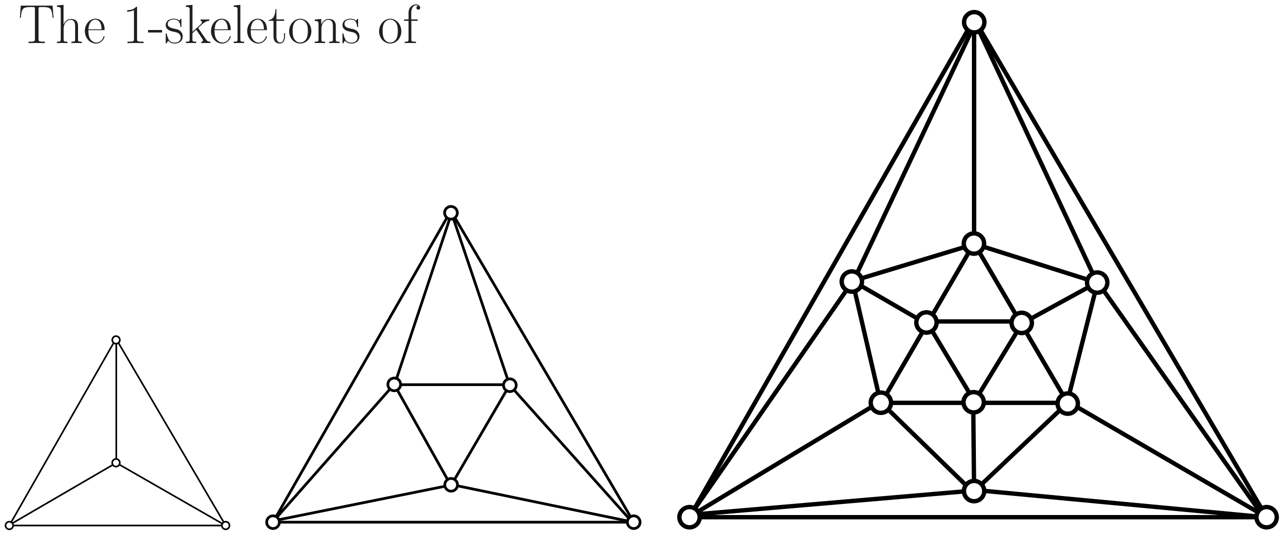
A graph is **locally** $\mathcal{C}$ if the neighbours of each vertex induce $\mathcal{C}$ (or a member of $\mathcal{C}$).

Lemma (A.J. 1994). Γ *distance-regular,* $k \leq 10$ *and locally* C_k*. Then* Γ *is*
– *one of the Platonic solids with* $\triangle$*'s as faces,*
– *Paley graph* $P(13)$*, Shrikhande graph,*
– *Klein graph (i.e., the* 3*-cover of* K_8*).*

Problem. Find a locally C_{15} distance-regular graph.

Platonic solids with $\triangle$'s as faces

The 1-skeletons of

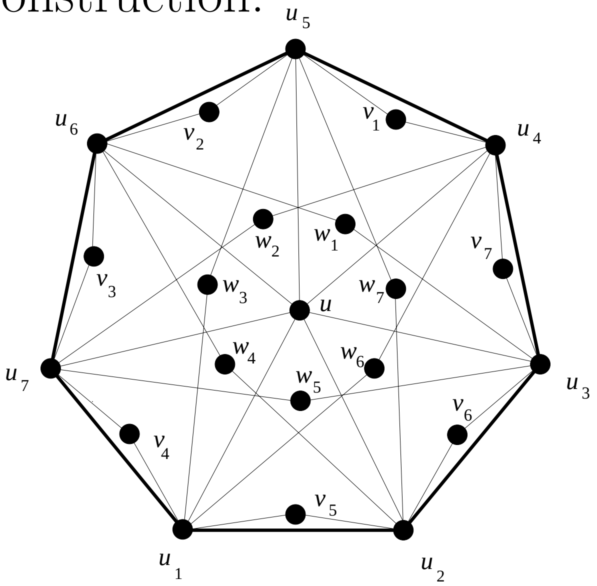


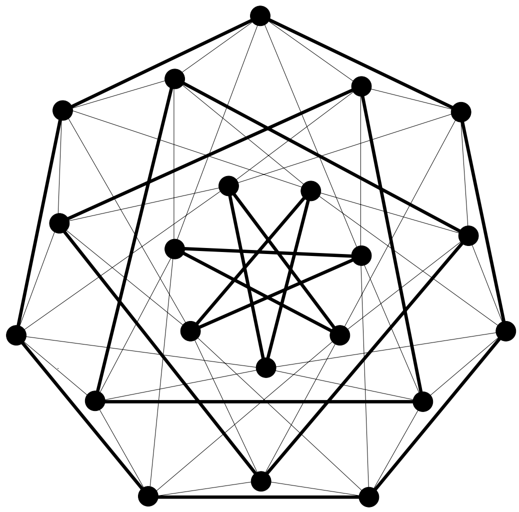
(a) the tetrahedron$= K_4$,

(b) the octahedron$= K_{2,2,2}$,

(c) the icosahedron.

There is only one feasible intersection array of distance-regular covers of K_8: $\{7,4,1;1,2,7\}$ - the Klein graph, i.e., the dual of the famous Klein map on a surface of genus 3. It must be the one coming from Mathon's construction.





Mathon's construction of an r-cover of K_{q+1}

A version due to Neumaier: using a subgroup K of the $\mathrm{GF}(q)^*$ of index r. For, let $q = rc + 1$ be a prime power and either c is even or $q - 1$ is a power of 2.

We use an equivalence relation $\mathcal{R}$ for $\mathrm{GF}(q)^2\backslash\{0\}$: $(v_1, v_2)\mathcal{R}(u_1, u_2)$ iff $\exists h \in K$ s.t. $(v_1h, v_2h) = (u_1, u_2)$.

vertices: equiv. classes vK, $v \in \mathrm{GF}(q)^2\backslash\{0\}$ of $\mathcal{R}$, and $(v_1, v_2)K{\sim}(u_1, u_2)K$ iff $v_1u_2 - v_2u_1 \in K$,

It is an antipodal distance-regular graph of diam. 3, with $r(q + 1) = (q^2 - 1)/c$ vertices, index r, $c_2 = c$ (vertex transitive, and also distance-transitive when r is prime and the char. of $\mathrm{GF}(q)$ is primitive mod r).

Theorem (Brouwer, 1983).
GQ(s,t) *minus a spread,* $t>1$
$\Longrightarrow$ $(s+1)$*-cover of* K_{st+1} *with* $c_2 = t-1$.

- good construction: q a prime power:

$$(s,t) = \begin{cases} (q,q), \\ (q-1,q+1), \\ (q+1,q-1), & \text{if } 2\,|\,q \\ (q,q^2). \end{cases}$$

- good characterization (geometric graphs),
- nonexistence

n	r	a_1	c_2	a cover Γ of K_n	#of Γ
5	3	1	1	L(Petersen)	1
6	2	2	2	Icosahedron	1
7	6	0	1	S_2(Hoffman-Singleton)	1
8	3	2	2	Klein graph	1
9	3	1	3	GQ$(2,4)\backslash$ spread	2
9	7	1	1	equivalent to the unique $PG(2,8)$	1
10	2	4	4	Johnson graph $J(6,3)$	1
10	4	2	2	GQ$(3,3)\backslash$ unique spread	≥ 1

n	r	a_1	c_2	a cover Γ of K_n	# of Γ
11	9	1	1	does not exist (PG(2,10))	0
12	5	2	2	Mathon's construction	≥ 1
13	11	1	1	open (PG(2,12))	?
14	2	6	6	equivalent to Paley graph $\{6,3;1,3\}$	1
14	3	4	4	Mathon's construction	≥ 1
14	6	2	2	Mathon's construction	≥ 1
16	2	6	8	[dCMM], [So] and [Th1]	1
16	2	8	6	unique two-graph, i.e., $\frac{1}{2}H(6,2)$	1
16	4	2	4	GQ(3,5)\ spread	≥ 5
16	6	4	2	GQ(5,3)\ spread	≥ 1
16	7	2	2	OPEN	?
16	8	0	2		≥ 1
17	3	5	5	Mathon's construction	≥ 1
17	5	3	3	GQ(4,4)\ unique spread	≥ 2
17	15	1	1	equivalent to PG(2,16), Mathon's construction	≥ 1
18	2	8	8	Mathon's construction	1
18	4	4	4	Mathon's construction	≥ 1
18	8	2	2	Mathon's construction	≥ 1
19	4	2	5	[Hae2] (GQ(3,6) does not exist	0
19	7	5	2	[Go4] (GQ(6,3) does not exist	0
19	17	1	1	open (PG(2,18))	?

Antipodal covers of diameter 4

Let Γ be an antipodal distance-regular graph of diameter 4, with v vertices, and let r be the size of its antipodal classes (we also use $\lambda := a_1$ and $\mu := c_2$).

The intersection array $\{b_0, b_1, b_2, b_3; c_1, c_2, c_3, c_4\}$ is determined by (k, a_1, c_2, r), and has the following form

$$\{k, k - a_1 - 1, (r - 1)c_2, 1; 1, c_2, k - a_1 - 1, k\},$$

A systematic approach:

– make a list of all small feasible parameters

– check also the Krein conditions and absolute bounds

Let $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$ be $\mathrm{ev}(\Gamma)$. The **antipodal quotient** is $\boldsymbol{\mathrm{SRG}(v/r, k, a_1, rc_2)}$, the old eigenvalues, i.e., $\theta_0 = k$, θ_2, θ_4, are the roots of

$$x^2 - (a_1 - rc_2)x - (k - rc_2) = 0$$

and the new eigenvalues, i.e., θ_1, θ_3, are the roots of

$$x^2 - a_1x - k = 0.$$

The following relations hold for the eigenvalues:

$$\theta_0 = -\theta_1\theta_3, \text{ and } (\theta_2 + 1)(\theta_4 + 1) = (\theta_1 + 1)(\theta_3 + 1).$$

The multiplicities are $m_0 = 1$, $m_4 = (v/r) - m_2 - 1$,

$$m_2 = \frac{(\theta_4 + 1)k(k - \theta_4)}{rc_2(\theta_4 - \theta_2)} \text{ and } m_{1,3} = \frac{(r-1)v}{r(2 + a_1\theta_{1,3}/k)}.$$

Parameters of the antipodal quotient can be expressed in terms of eigenvalues and r: $\quad k = \theta_0$,

$$a_1 = \theta_1 + \theta_3,\ b_1 = -(\theta_2 + 1)(\theta_4 + 1),\ c_2 = \frac{\theta_0 + \theta_2\theta_4}{r}.$$

The eigenvalues θ_2, θ_4 are integral, $\theta_4 \le -2,\ 0 \le \theta_2$, with $\theta_2 = 0$ iff Γ is bipartite.

Furthermore, $\theta_3 < -1$, and the eigenvalues θ_1, θ_3 are integral when $a_1 \neq 0$.

We define for $s \in \{0,1,2,3,4\}$ the symmetric 4×4 matrix $P(s)$ with its ij-entry being equal to $p_{ij}(s)$.

For $b_1 = k - 1 - \lambda$, $k_2 = rkb_1/\mu$, $a_2 = k - \mu$ and $b_2 = (r-1)\mu/r$ we have

$$P(0) = \begin{pmatrix} k & 0 & 0 & 0 \\ & k_2 & 0 & 0 \\ & & (r-1)k & 0 \\ & & & r-1 \end{pmatrix},$$

$$P(1) = \begin{pmatrix} \lambda & b_1 & 0 & 0 \\ & k_2 - b_1 r & b_1(r-1) & 0 \\ & & \lambda(r-1) & r-1 \\ & & & 0 \end{pmatrix},$$

$$P(2) = \begin{pmatrix} \mu/r & a_2 & b_2 & 0 \\ & k_2 - r(a_2+1) & (r-1)(k-\mu) & r-1 \\ & & b_2(r-1) & 0 \\ & & & 0 \end{pmatrix},$$

$$P(3) = \begin{pmatrix} 0 & b_1 & \lambda & 1 \\ & k_2 - rb_1 & b_1(r-1) & 0 \\ & & \lambda(r-2) & r-2 \\ & & & 0 \end{pmatrix},$$

$$P(4) = \begin{pmatrix} 0 & 0 & k & 0 \\ & k_2 & 0 & 0 \\ & & k(r-2) & 0 \\ & & & r-2 \end{pmatrix}.$$

The matrix of eigenvalues $P(\Gamma)$ (with $\omega_j(\theta_i)$ being its ji-entry) has the following form:

$$P(\Gamma) = \begin{pmatrix} 1 & \theta_0 & \theta_0 b_1/c_2 & \theta_0(r-1) & r-1 \\ 1 & \theta_1 & 0 & -\theta_1 & -1 \\ 1 & \theta_2 & -r(\theta_2+1) & \theta_2(r-1) & r-1 \\ 1 & \theta_3 & 0 & -\theta_3 & -1 \\ 1 & \theta_4 & -r(\theta_4+1) & \theta_4(r-1) & r-1 \end{pmatrix}$$

Theorem. (JK 1995).

Γ antipodal distance-regular graph, diam 4, and eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$.

Then $q^2_{11},\ q^3_{12},\ q^4_{13},\ q^2_{22},\ q^4_{22},\ q^3_{23},\ q^4_{24},\ q^4_{33} > 0$,
$r = 2$ *iff* $q^1_{11} = 0$ *iff* $q^3_{11} = 0$ *iff* $q^3_{13} = 0$ *iff* $q^3_{33} = 0$,
$q^2_{12} = q^4_{12} = q^4_{14} = q^3_{22} = q^4_{23} = q^4_{34} = 0$ *and*

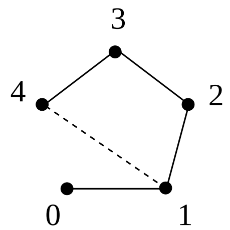
(i) $(\theta_4 + 1)^2(k^2 + \theta_2^3) \geq (\theta_2 + 1)(k + \theta_2\theta_4)$, *with equality iff* $q^2_{22} = 0$,

(ii) $(\theta_2 + 1)^2(k^2 + \theta_4^3) \geq (\theta_4 + 1)(k + \theta_2\theta_4)$, *with equality iff* $q^4_{44} = 0$,

(iii) $\theta_3^2 \geq -\theta_4$, *with equality iff* $q^4_{11} = 0$.

Let E be a primitive idempotent of a distance-regular graph of diameter d. The **representation diagram** $\mathbf{\Delta}_{\boldsymbol{E}}$ is the undirected graph with vertices $0, 1, \ldots d$, where we join two distinct vertices i and j whenever $q_{ij}^s = q_{ji}^s \neq 0$.

Recall Terwilliger's characterization of Q-polynomial association schemes that a d-class association scheme is Q-polynomial iff the representation diagram a minimal idempotent, is a path. For $s = 1$ and $r = 2$ we get the following graph:



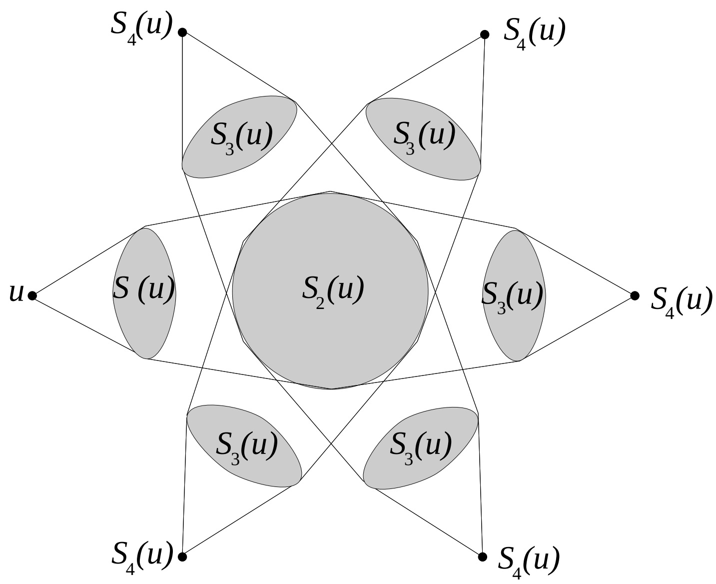
Based on the above information we have:

Corollary. Γ *antipodal, distance-regular graph with diam.* 4. *TFAE*

(i) Γ *is* Q*-polynomial.*

(ii) $r = 2$ *and* $q_{11}^4 = 0$.

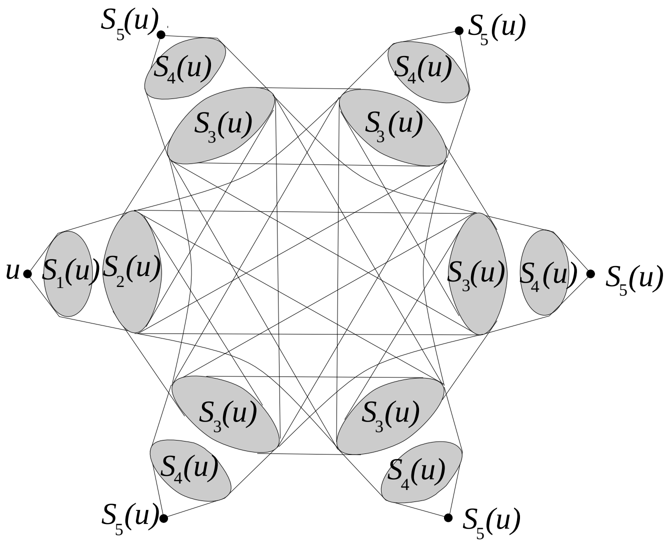
Suppose (i)-(ii) hold, then θ_0, θ_1, θ_2, θ_3, θ_4 *is a unique* Q*-polynomial ordering, and* $q_{ij}^h = 0$ *when* $i + j + h$ *is odd, i.e., the* Q*-polynomial structure is dual bipartite.*



An antipodal distance-regular graph of diameter 4
(the distance partition corresponding to an antipodal class).

#	Γ	n	k	λ	μ	H	r	$r.n$
1	! Folded 5-cube	16	5	0	2	! Wells graph	2	32
2	! $\overline{T(6)}$	15	6	1	3	! 3.Sym(6).2	3	45
3	! $\overline{T(7)}$	21	10	3	6	! 3.Sym(7)	3	63
4	folded $J(8,4)$	35	16	6	8	! Johnson graph $J(8,4)$	2	70
5	! truncated 3-Golay code	81	20	1	6	shortened 3-Golay code	3	243
6	! folded halved 8-cube	64	28	12	12	! halved 8-cube	2	128
7	$S_2(S_2(McL.))$	105	32	4	12	S_2(Soicher1 graph)	3	315
8	Zara graph (126,6,2)	126	45	12	18	$3.O_6^-(3)$	3	378
9	! S_2(McLaughlin graph) [Br3]	162	56	10	24	! Soicher1 graph	3	486
10	hyperbolic pts. of $PG(6,3)$	378	117	36	36	$3.O_7(3)$	3	1134
11	Suzuki graph	1781	416	100	96	Soicher2 [Soi]	3	5346
12		306936	31671	3510	3240	$3.Fi_{24}^-$	3	

Non-bipartite antipodal distance-regular graphs of diameter 4.



#	Γ	n	k	λ	μ	H	r	t	$r.n$
1	! Petersen graph	10	3	0	1	! Dodecahedron	2	1	20
2	3-Golay code	243	22	1	2	short. ext. 3-Golay code	3	9	729
3	folded Johnson graph $J(10{,}5)$	126	25	8	8	! Johnson graph $J(10{,}5)$	2	9	252
4	folded halved 10-cube	256	45	16	6	! halved 10-cube	2	15	512

Non-bipartite antipodal distance-regular graphs of diameter 5.

VIII. 1-homogeneous graphs

- a homogemeous property
- examples
- a local approach and the CAB property
- recursive relations on parameters
- algorithm
- a classification of Terwilliger graphs
- modules

$\boxed{\textbf{Homogeneous property}}$

(in the sense of Nomura)

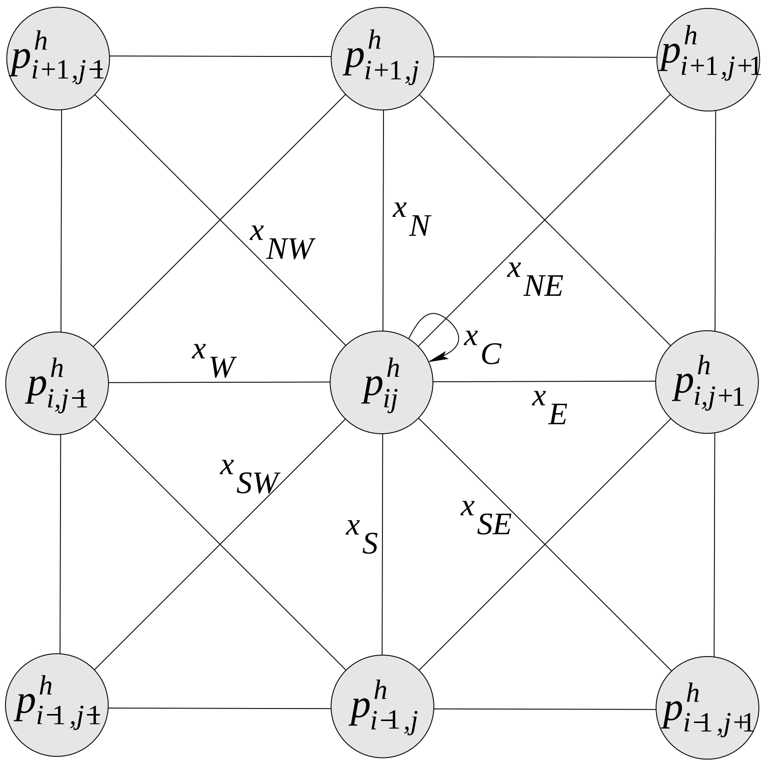
Γ graph, diameter d, $x, y \in V(\Gamma)$, s.t. $\partial(x,y) = h$, $i, j \in \{0, \ldots, d\}$. Set $\boldsymbol{D_i^j = D_i^j(x,y)} := \Gamma_i(x) \cap \Gamma_j(y)$ and note $|D_i^j| = p_{ij}^h$.

The graph Γ is ***h*-homogeneous** when the partition

$$\{D_i^j \mid 0 \le i, j \le d,\ D_i^j \ne \emptyset\}$$

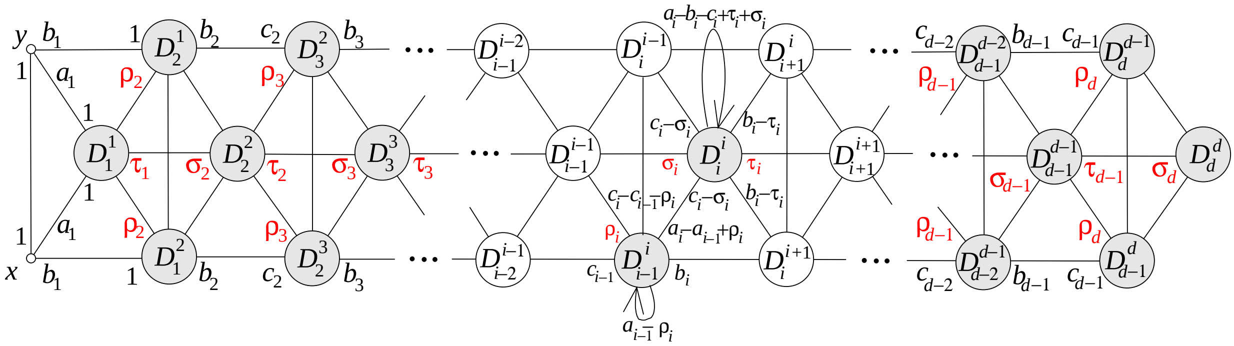
is *equitable* for every $x, y \in V(\Gamma)$, $\partial(x,y) = h$, and the parameters corresponding to equitable partitions are *independent* of x and y.

0-homogeneous $\Longleftrightarrow$ distance-regular



$$x_{SW} + x_S + x_{SE} = c_i,\ x_W + x_C + x_E = a_i,\ x_{NW} + x_N + x_{NE} = b_i,$$

$$x_{NW} + x_W + x_{SW} = c_j,\ x_N + x_C + x_S = a_j,\ x_{NE} + x_E + x_{SE} = b_j.$$



For $i \in \{1, \dots, d\}$

$$|D_{i-1}^i| = |D_i^{i-1}| = \frac{b_1 b_2 \dots b_{i-1}}{c_1 c_2 \dots c_{i-1}}, \quad |D_i^i| = a_i \frac{b_1 b_2 \dots b_{i-1}}{c_1 c_2 \dots c_i},$$

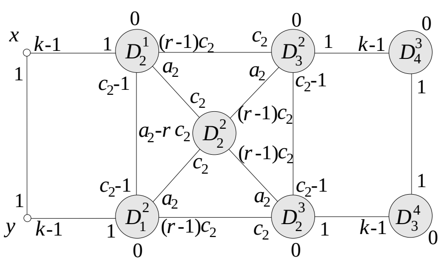
and therefore $D_i^{i-1} \neq \emptyset \neq D_{i-1}^i$.

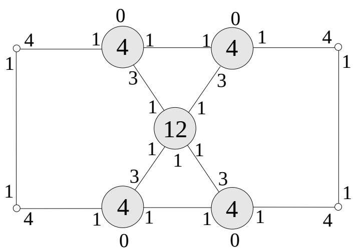
A distance-regular graph Γ is **1-homogeneous** when the distance distribution corresponding to an edge is equitable.

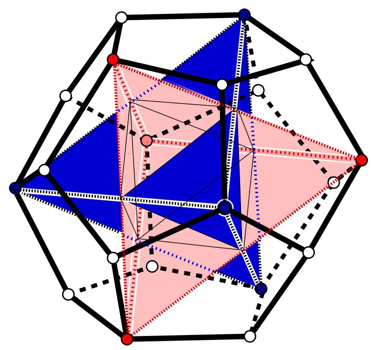
Some examples of 1-homogeneous graphs

distance-regular graphs with at most one i, s.t. $a_i \neq 0$:

– bipartite graphs,
– generalized Odd graphs;

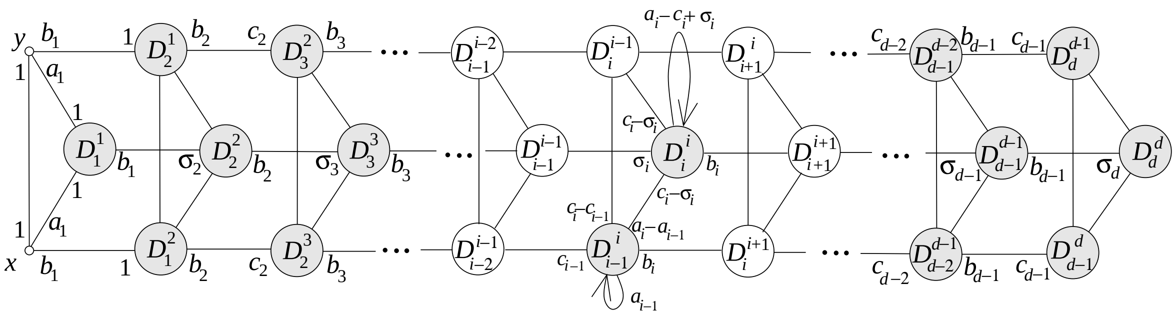






the Wells graph

A 1-homogeneous graph Γ of diameter $d \geq 2$ and $a_1 \neq 0$ is locally disconnected iff it is a regular near $2d$-gon (i.e., a distance-regular graph with $a_i = c_i a_1$ and no induced $K_{1,2,1}$).

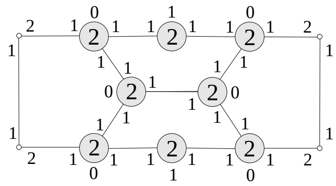


If Γ is locally disconnected, then for $i = 1, \ldots, d-1$.

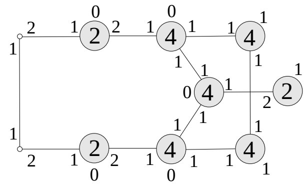
$$\tau_i = b_i \quad \text{and} \quad \sigma_{i+1} = \frac{c_{i+1} a_i}{a_{i+1}}.$$

Some examples of 1-homo. graphs, cont.

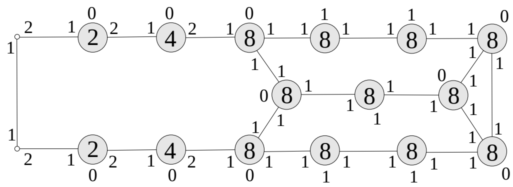
- the Taylor graphs,
- the Johnson graph $J(2d, d)$,
- the folded Johnson graph $\overline{J}(4d, 2d)$,
- the halved n-cube $H(n, 2)$,
- the folded halved $(2n)$-cube,
- cubic distance-regular graphs.



the dodecahedron



the Coxeter graph



the Biggs-Smith graph

The **local graph $\Delta(x)$** is the subgraph of Γ induced by the neighbours of x. It has k vertices & valency a_1.

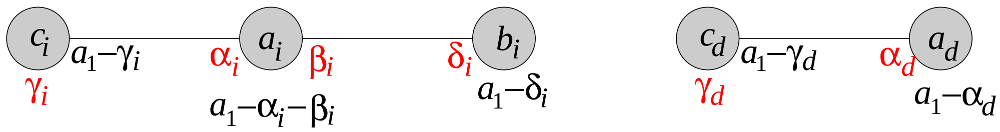
All local graphs of a **1-homogeneous** graph are

(i) connected strongly regular graphs
with the same parameters, or

(ii) disjoint unions of (a_1+1)-cliques.

A local approach

For $x, y \in V(\Gamma)$, s.t. $\partial(x, y) = i$, let $\mathbf{CAB}_i(x, y)$ be the partition $\{C_i(x, y), A_i(x, y), B_i(x, y)\}$ of $\Gamma(y)$.



Γ has the **$\mathbf{CAB}_j$ property**, if $\forall i \le j$ the partition $\mathrm{CAB}_i(x, y)$ is equitable $\forall x, y \in V(\Gamma)$, s.t. $\partial(x, y) = i$.

the CAB_1 property $\Longleftrightarrow$ Γ is locally strongly regular

Theorem [JK'00]. Γ *drg, diam.* d, $a_1 \neq 0$. *Then* Γ *is* 1-*homogeneous* $\Longleftrightarrow$ Γ *has the CAB property.*

A two way counting gives us for $i = 2, \ldots, d$:

$$\alpha_i c_{i-1} = \sigma_i \, \alpha_{i-1},$$

$$\beta_{i-1} b_i = \tau_{i-1} \, \beta_i,$$

$$\gamma_i(c_{i-1} - \sigma_{i-1}) = \rho_i \, \alpha_{i-1}.$$

The quotient matrices corresponding to CAB_i partitions are, for $1 \le i \le j$, $i \ne d$,

$$Q_i = \begin{pmatrix} \gamma_i & a_1 - \gamma_i & 0 \\ \alpha_i & a_1 - \beta_i - \alpha_i & \beta_i \\ 0 & \delta_i & a_1 - \delta_i \end{pmatrix},$$

and when $j = d$ also $Q_d = \begin{pmatrix} \gamma_d & a_1 - \gamma_d \\ \alpha_d & a_1 - \alpha_d \end{pmatrix}$,

if $a_d \ne 0$, and $Q_d = (\gamma_d)$, if $a_d = 0$

Let Γ be a 1-homogeneous graph with diameter d that is locally connected and let $\delta_0 := 0$.

Then $a_i \neq 0$, $a_1 - \gamma_i \neq 0$, and we have the following recursion: $\gamma_i = \delta_{i-1}$,

$$\alpha_i = \frac{(a_1 - \delta_{i-1})c_i}{a_i}, \qquad \delta_i = \frac{a_i \mu'}{a_1 - \delta_{i-1}}, \qquad \beta_i = b_i \delta_i / a_i,$$

for $i \in \{1, 2, \ldots, d-1\}$, and when $i = d$

$\gamma_d = \delta_{d-1}$, $\alpha_d = (a_1 - \delta_{d-1})c_d/a_d$, if $a_d \neq 0$,

and $\gamma_d = a_1$, if $a_d = 0$.

An bf algorithm to calculate all possible intersection arrays of 1-homogeneous graphs for which we know that local graphs are connected SRGs with given parameters,

Given the parameters (k', λ', μ') of a connected SRG, calculate its eigenvalues $k' = a_1 > p > q$ and

$$k = v' = \frac{(a_1 - p)(a_1 - q)}{a_1 + pq}, \quad b_1 = k - a_1 - 1, \quad \alpha_1 = 1,$$

$$\beta_1 = a_1 - \lambda' - 1, \qquad \gamma_1 = 0, \qquad \delta_1 = \mu'.$$

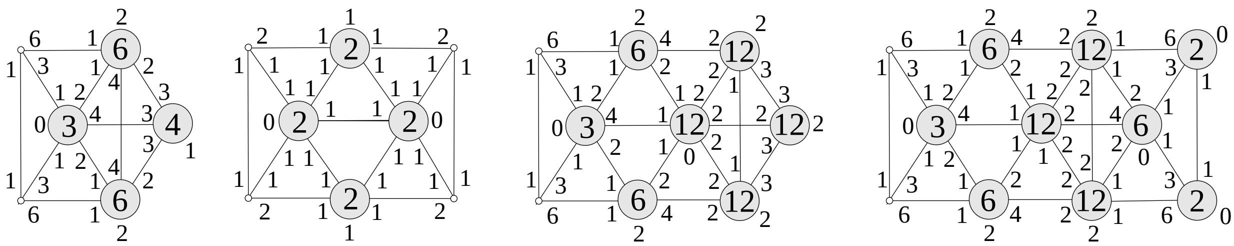
and initialize the sets $F := \emptyset$ (final), $N := \emptyset$ (new) and $S := \{\{k, b_1, \delta_1\}\}$ (current).

for $i \geq 2$ **and** $S \neq \emptyset$ **do**
  for $\{c_2, \ldots, c_{i-1}, \delta_{i-1}; k, b_1, \ldots, b_{i-1}\} \in S$ **do**
    $\gamma_i := \delta_{i-1}$;
    if $\gamma_i = a_1$ **then** $a_i = 0; c_i = k;\ F := F \cup \{\{k, b_1, \ldots, b_{i-1}; 1, c_2, c_3, \ldots, c_i\}\}$ **fi**;
    if $\gamma_i < a_1$ **then**
      assume diameter $= i$ and calculate α_i, a_i, c_i
      if $\Big(k_i \in \mathbb{N}$ **and** $\alpha_i,\ a_i,\ c_i \in \mathbb{N}$ **and** $a_i(a_1 - \alpha_i)/2,\ c_i\gamma_i/2 \in \mathbb{N}_0\Big)$
      then $F := F \cup \{\{k, b_1, \ldots, b_{i-1}; 1, c_2, \ldots, c_i\}\}$ **fi**;
      assume diameter $> i$;
      for $c_i = \max(c_{i-1}, \gamma_i) + 1, \ldots, b_1$ **do**
        calculate $\alpha_i, \beta_i, \delta_i, b_i, a_i$
        if $\Big(k_i \in \mathbb{N}$ **and** $\alpha_i,\ \beta_i,\ \delta_i,\ b_i,\ a_i \in \mathbb{N}$ **and** $\delta_i \geq \gamma_i$
          and $\dfrac{c_i\gamma_i}{2},\ \dfrac{(a_1 - \beta_i - \alpha_i)a_i}{2},\ \dfrac{b_i(a_1 - \delta_i)}{2} \in \mathbb{N}_0\Big)$
        then $N := N \cup \{\{c_2, \ldots, c_i, \delta_i; k, b_1, \ldots, b_i\}\}$ **fi**;
      od;
    fi;
  od;
  $S := N;\quad N := \emptyset$;
od;

Locally Moore graphs

Theorem [JK'00]. *A graph whose local graphs are Moore graphs is* 1*-homogeneous iff it is one of the following graphs:*

- *the icosahedron* ($\{5, 2, 1; 1, 2, 5\}$),
- *the Doro graph* ($\{10, 6, 4; 1, 2, 5\}$),
- *the Conway-Smith graph* ($\{10, 6, 4, 1; 1, 2, 6, 10\}$),
- *the compl. of* $T(7)$ ($\{10, 6; 1, 6\}$).



Terwilliger graphs

A connected graph with diameter at least two is called a **Terwilliger graph** when every μ-graph has the same number of vertices and is complete.

A distance-regular graph with diameter $d \geq 2$ is a Terwilliger graph iff it contains no induced C_4.

Corollary [JK'00]. *A Terwilliger graph with $c_2 \geq 2$ is 1-homogeneous iff it is one of the following graphs:*
(i) the icosahedron,
(ii) the Doro graph,
(iii) the Conway-Smith graph.

Modules

Γ distance-regular, diam. $d \geq 2$. Let x and y be adjacent vertices and $D_i^j = D_i^j(x,y)$.
Suppose $a_1 \neq 0$. Then for $i \neq d$, $a_i \neq 0$, i.e., $D_i^i \neq \emptyset$. Moreover, $D_d^d = \emptyset$ iff $a_d = 0$.

Let w_{ij} be a characteristic vector of the set D_i^j and $W = W(x,y) := \mathrm{Span}\{w_{ij} \mid i,j = 0,\ldots,d\}$. Then

$$\dim W = \begin{cases} 3d & \text{if } a_d \neq 0, \\ 3d-1 & \text{if } a_d = 0. \end{cases}$$

For $\forall xy \in E\Gamma$, we define the scalar $f = f(x, y)$:

$$f = \frac{1}{a_1}\Big|\{(z, w) \in X^2 \mid z, w \in \Gamma(x, y),\ \partial(z, w) = 2\}\Big|.$$

f is the average degree of the complement of the λ-graph. Then $0 \leq f \leq a_1 - 1, b_1$ and for $\theta \in \mathrm{ev}(\Gamma)$, $E = E(\theta)$ the Gram matrix of $E\hat{x}$, $E\hat{y}$, w_{11} is

$$\frac{m_\theta^3}{n} \det \begin{pmatrix} \omega_0 & \omega_1 & a_1\omega_1 \\ \omega_1 & \omega_0 & a_1\omega_1 \\ a_1\omega_1 & a_1\omega_1 & c \end{pmatrix}$$

where $c = a_1\big(\omega_0 + (a_1 - f - 1)\omega_1 + f\omega_2\big)$.

So

$$(\omega - \omega_2)(1 + \omega)f \le (1 - \omega)(a_1\omega + 1 + \omega),$$

i.e.,

$$\boxed{(k + \theta)(1 + \theta)\ f \ \le\ b_1\big(k + \theta(a_1 + 1)\big).}$$

We now consider which of $\theta_1, \theta_2, \ldots, \theta_d$ gives the best bounds for f. Let θ denote one of $\theta_1, \theta_2, \ldots, \theta_d$, and assume $\theta \ne -1$. If $\theta > -1$ (resp. $\theta < -1$), the obtained inequality gives an upper (resp. lower) bound for f.

Consider the partial fraction decompostion

$$b_1\frac{k+\theta(a_1+1)}{(k+\theta)(1+\theta)} \quad = \quad \frac{b_1}{k-1}\Big(\frac{ka_1}{k+\theta}+\frac{b_1}{1+\theta}\Big).$$

Since the map $F:\mathbb{R}\setminus\{-k,-1\}\longrightarrow\mathbb{R}$, defined by

$$x\mapsto\frac{ka_1}{k+x}+\frac{b_1}{1+x}$$

is strictly decreasing on the intervals $(-k,-1)$ and $(-1,\infty)$, we find that the least upper bound for f is obtained at $\theta=\theta_1$, and and the greatest lower bound is obtained at $\theta=\theta_d$:

$$\boxed{b_1\frac{k+\theta_d(a_1{+}1)}{(k{+}\theta_d)(1{+}\theta_d)}\leq f\leq b_1\frac{k+\theta_1(a_1{+}1)}{(k{+}\theta_1)(1{+}\theta_1)}.}$$

Set $H = H(x,y) := \mathrm{Span}\{\hat{x}, \hat{y}, w_{11}\}$

Suppose Γ **is 1-homogeneous**. So $AW = W$. The Bose-Mesner algebra $\mathcal{M}$ is generated by A, so also $\mathcal{M}W = W = \mathcal{M}H$ $(:=\mathrm{Span}\{mh \,|\, m \in \mathcal{M}, h \in H\})$.

$E_0, E_1, \ldots, E_d$ is a basis for $\mathcal{M}$, so $E_iE_j = \delta_{ij}E_i$ and

$$\mathcal{M}H = \sum_{i=0}^{d} E_iH \qquad \text{(direct sum)},$$

Note $\dim(E_0H) = 1$ and $3 \geq \dim(E_iH) \geq 2$, and $\dim(E_iH) = 2$ implies $i \in \{1, d\}$.

If $t := |\{i \,|\, \dim(E_iH) = 2\}|$, then $t \in \{0,1,2\}$ and $\dim(\mathcal{M}H) = 3d + 1 - t$. Hence $t = 2$ when $a_d = 0$.

IX. Tight distance-regular graphs

- alternative proof of the fundamental bound
- definition
- characterizations
- examples
- parametrization
- AT4 family
- complete multipartite μ-graphs
- classifications of AT4(qs, q, q) family
- uniqueness of the Patterson graph
- locally GQ

Lemma. *Let $\Gamma =$ be a k-regular, connected graph on n vertices, e edges and t triangles, with eigenvalues*

$$k = \eta_1 \leq \eta_2 \leq \cdots \leq \eta_n.$$

Then

(i) $\sum_{i=1}^{n} \eta_i = 0,$

(ii) $\sum_{i=1}^{n} \eta_i^2 = nk = 2e,$

(iii) $\sum_{i=1}^{n} \eta_i^3 = nk\lambda = 6t,$

if λ is the number of triangles on every edge.

Now suppose that r and s are resp. an upper and lower bounds on the nontrivial eigenvalues.
Hence $(\eta_i - r)(\eta_i - s) \leq 0$ for $i \neq 1$, and so

$$\sum_{i=2}^{k} (\eta_i - s)(\eta_i - r) \leq 0,$$

which is equivalent to

$$n(k + rs) \leq (k - s)(k - r).$$

Equality holds if and only if

$$\eta_i \in \{r, s\} \text{ for } i = 2, \ldots, n,$$

i.e., Γ is strongly regular with eigenvalues k, r and s.

Let us rewrite this for a local graph of a vertex of a distance-regular graph:

$$k \leq \frac{(a_1 - b^-)(a_1 - b^+)}{a_1 + b^- b^+}.$$

where b^- and b^+ are the lower and the upper bound for the nontrivial eigenvalues of the local graph.

We define for a distance-regular graph with diam. d and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$

$$b^- = -1 - \frac{b_1}{\theta_1 + 1} \quad \text{and} \quad b^+ = -1 - \frac{b_1}{\theta_d + 1},$$

and note $b^- < 0$ and $b^+ > 0$.

Theorem [Terwilliger]. *Let x be a vertex of a distance-regular graph Γ with diameter $d \geq 3$, $a_1 \neq 0$ and let*

$$a_1 = \eta_1 \geq \eta_2 \geq \ldots \geq \eta_k$$

be the eigenvalues of the local graph $\Delta(x)$. Then,

$$b^+ \geq \eta_2 \geq \eta_k \geq b^-.$$

Proof. Let us define N_1 to be the adjacency matrix of the local graph $\Delta = \Delta(x)$ for the vertex x and let N to be the Gram matrix of the normalized representations of all the vertices in Δ.

Since Γ is not complete multipartite, we have $\omega_2 \neq 1$ and

$$\begin{aligned} N &= I_k + N_1\omega_1 + (J_k - I_k - N_1)\omega_2 \\ &= (1-\omega_2)\left(I_k + N_1\frac{\omega_1-\omega_2}{1-\omega_2} + J_k\frac{\omega_2}{1-\omega_2}\right). \end{aligned}$$

The matrix $N/(1-\omega_2)$ is positive semi-definite, so its eigenvalues are nonegative and we have for $i = 2, \ldots, k$:

$$1 + \frac{\omega_1-\omega_2}{1-\omega_2}\eta_i \geq 0, \quad \text{i.e.,} \quad 1 + \frac{1+\theta}{\theta+b_1+1}\eta_i \geq 0.$$

Since k is the spectral radious, by the expression for $1-\omega_2$, we have $\theta > -b_1 - 1$ and thus also

$$(1+\theta)\eta_i \geq -(\theta + b_1 + 1).$$

If $\theta > -1$, then

$$\eta_i \geq -\frac{\theta + b_1 + 1}{\theta + 1} = -1 - \frac{b_1}{\theta + 1}.$$

The expression on the RHS is an increasing function, so it is uper-bounded by b^-.

Similarly if $\theta < -1$, then η_i is lower-bounded by b^+. ■

Fundamental bound (FB) [JKT'00]

Γ distance-regular, diam. $d \geq 2$,
and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$.

$$\boxed{\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \geq \frac{-ka_1b_1}{(a_1+1)^2}}$$

If equality holds in the FB and Γ is nonbipartite, then Γ is called a **tight graph**.

For $d{=}2$ we have $b_1 = -(1+\theta_1)(1+\theta_2)$, $b^+ = \theta_1$, $b^- = \theta_2$, and thus **Γ is tight** (i.e., $\theta_1 = 0$) **iff $\Gamma = K_{t\times n}$ with $t > 2$** (i.e., $a_1 \neq 0$ and $\mu = k$).

Characterizations of tight graphs

Theorem [JKT'00]. *A nonbipartite distance-regular graph* Γ *with diam.* $d \geq 3$ *and eigenvalues* $\theta_0 > \theta_1 > \cdots > \theta_d$. *TFAE*

(i) Γ *is tight.*

(ii) Γ *is* 1*-homogeneous and* $a_d = 0$.

(iii) *the local graphs of* Γ *are connected strongly regular graphs with eigenvalues* a_1, b^+, b^-, *where*

$$b^- = -1 - \frac{b_1}{\theta_1 + 1} \quad \text{and} \quad b^+ = -1 - \frac{b_1}{\theta_d + 1}.$$

Examples of tight graphs

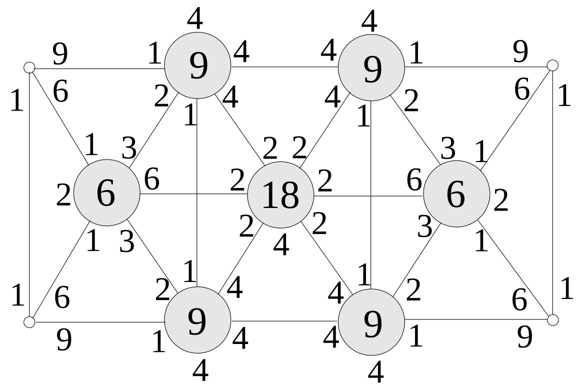
- the Johnson graph $J(2d, d)$,
- the halved cube $H(2d, 2)$,
- the Taylor graphs,
- the AT4 family
  (antipodal tight DRG with diam. 4),
- the Patterson graph $\{280, 243, 144, 10; 1, 8, 90, 280\}$
  (related to the sporadic simple group of Suzuki).

(i) The **Johnson graph $J(2d,d)$** has diameter d and intersection numbers

$$a_i = 2i(d-i), \quad b_i = (d-i)^2, \quad c_i = i^2 \quad (i = 0, \dots, d).$$

It is distance-transitive, antipodal double-cover and Q-polynomial with respect to θ_1.

Each local graph is a **lattice graph $K_d \times K_d$**, with parameters $(d^2, 2(d-1), d-2, 2)$ and nontrivial eigenvalues $r = d-2$, $s = -2$.

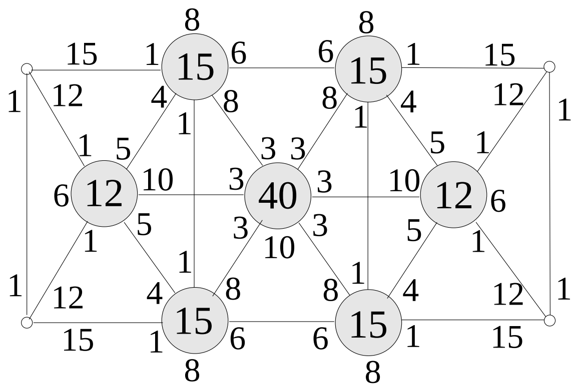


(ii) The **halved cube $H(2d,2)$** has diameter d and intersection numbers ($i=0,\dots d$)

$$a_i = 4i(d-i),\ \ b_i = (d-i)(2d-2i-1),\ \ c_i = i(2i-1).$$

It is distance-transitive, antipodal double-cover and Q-polynomial with respect to θ_1.

Each local graph is a **Johnson graph $J(2d,2)$**, with parameters $(d(2d-1), 4(d-1), 2(d-1), 4)$ and nontrivial eigenvalues $r = 2d-4$, $s=-2$.



(iii) The **Taylor graphs** are the double-covers of complete graphs, i.e., distance-regular graphs with intersection arrays $\{k, c_2, 1; 1, c_2, k\}$. They have diameter 3, and are Q-polynomial with respect to both θ_1, θ_d, given by $\theta_1 = \alpha$, $\theta_d = \beta$, where

$$\alpha + \beta = k - 2c_2 - 1, \quad \alpha\beta = -k, \quad \text{and} \quad \alpha > \beta.$$

Each local graph is strongly-regular with parameters (k, a_1, λ, μ), where $a_1 = k - c_2 - 1$,

$$\lambda = \frac{3a_1 - k - 1}{2}, \ \mu = \frac{a_1}{2}, \ r = \frac{\alpha - 1}{2} \text{ and } s = \frac{\beta - 1}{2}.$$

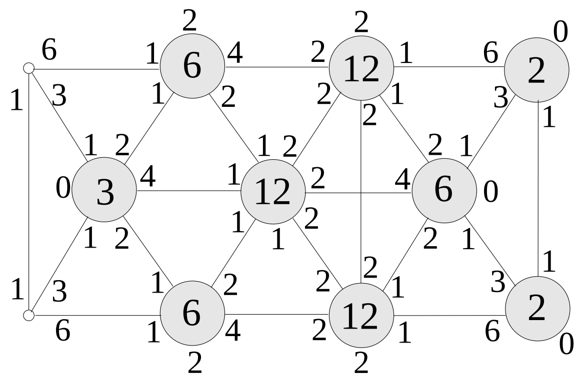
We note both a_1, c_2 are even and k is odd.

For example, the local graphs of the double-cover of K_{18} with $c_2 = 8$ are the **Paley graphs $\boldsymbol{P(17)}$**.

(iv) The **Conway-Smith graph, 3.$\boldsymbol{Sym}$(7)** has intersection array $\{10, 6, 4, 1; 1, 2, 6, 10\}$ and can be obtained from a sporadic Fisher group.

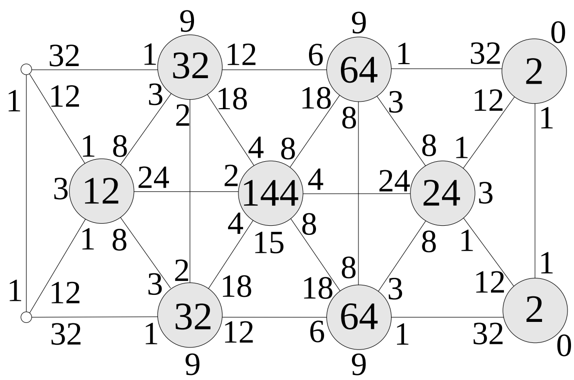
It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is a **Petersen graph**, with parameters $(10, 3, 0, 1)$ and nontrivial eigenvalues $r = 1,\ s = -2$.



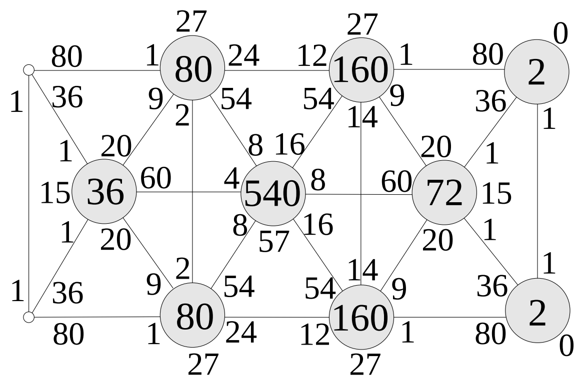
(v) The **$3.O_6^-(3)$-graph** has intersection array $\{45, 32, 12, 1; 1, 6, 32, 45\}$ and can be obtained from a sporadic Fisher group. It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is a **generalized quadrangle $GQ(4,2)$**, with parameters $(45, 12, 3, 3)$ and nontrivial eigenvalues $r = 3$, $s = -3$.



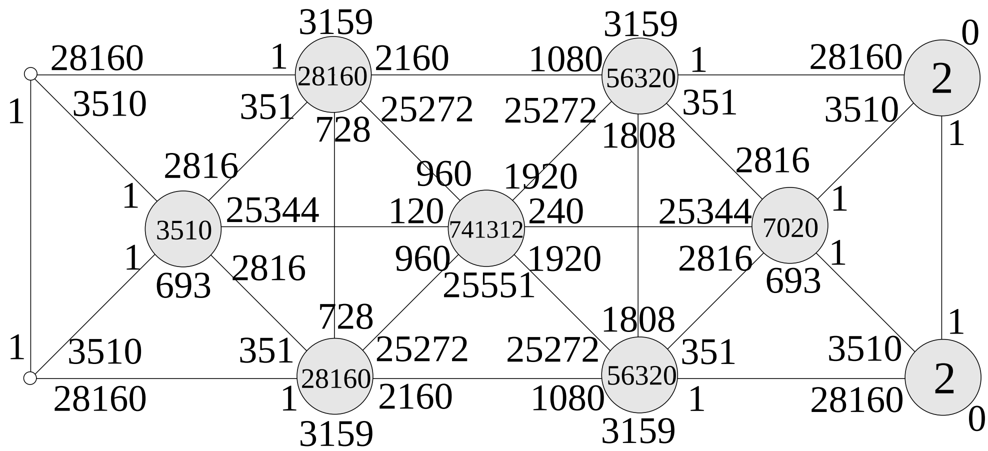
(vi) The **3.O_7(3)-graph** has intersection array $\{117, 80, 24, 1; 1, 12, 80, 117\}$ and can be obtained from a sporadic Fisher group. It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is strongly-regular with parameters $(117, 36, 15, 9)$, and nontrivial eigenvalues $r = 9$, $s = -3$.



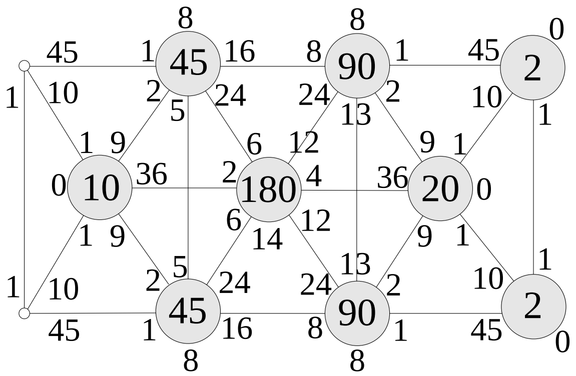
(vii) The **$3.Fi_{24}^-$-graph** has intersection array $\{31671, 28160, 2160, 1; 1, 1080, 28160, 31671\}$ and can be obtained from Fisher groups.
It is distance-transitive, antipodal 3-cover and is not Q-polynomial.

Each local graph is **SRG(31671,3510,693,351)** and $r = 351$, $s = -9$. They are related to Fi_{23}.



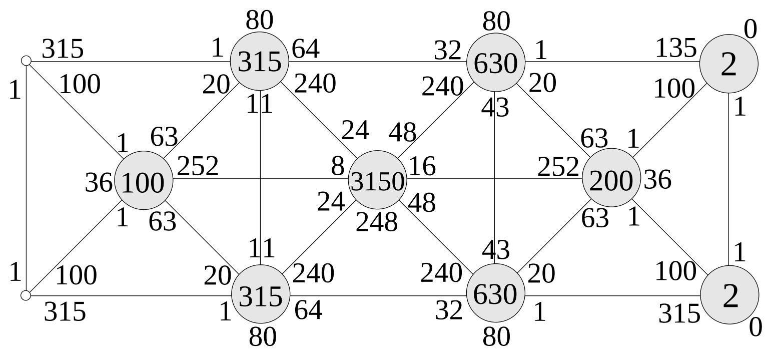
(viii) The **Soicher1 graph** has intersection array $\{56, 45, 16, 1; 1, 8, 45, 56\}$. It is antipodal 3-cover and is not Q-polynomial.

Each local graph is the **Gewirtz graph** with parameters $(56, 10, 0, 2)$ and $r = 2$, $s = -4$.



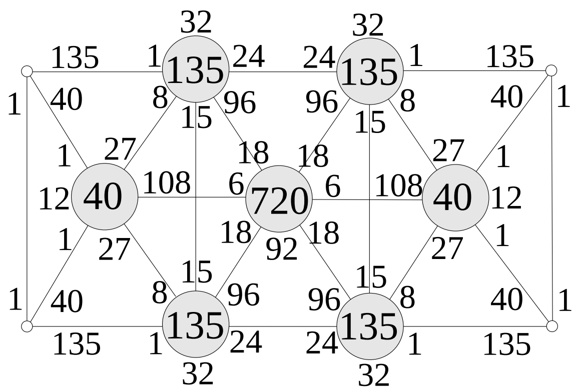
(ix) The **Soicher2 graph** has intersection array $\{416, 315, 64, 1; 1, 32, 315, 416\}$. It is antipodal 3-cover and is not Q-polynomial.

Each local graph is **SRG(416,100,36,20)** and $r = 20$, $s = -4$.



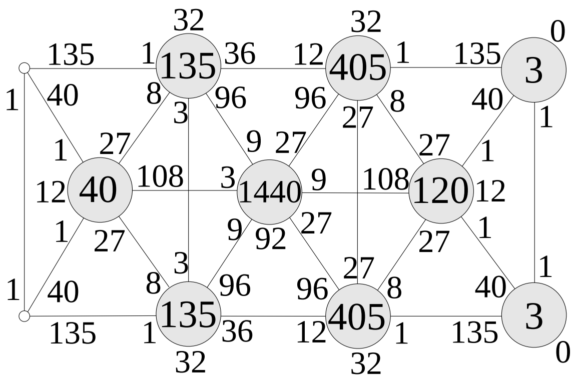
(x) The **Meixner1 graph** has intersection array $\{176, 135, 24, 1; 1, 24, 135, 176\}$. It is antipodal 2-cover and is Q-polynomial.

Each local graph is **SRG(176,40,12,8)** and $r = 8$, $s = -4$.



(xi) The **Meixner2 graph** has intersection array $\{176, 135, 36, 1; 1, 12, 135, 176\}$. It is antipodal 4-cover and is distance-transitive.

Each local graph is **SRG(176,40,12,8)** and $r = 8$, $s = -4$.



Theorem [JKT'00]. Γ *dr, diam.* $d \geq 3$.
Let θ, θ' *be a permutation of* θ_1, θ_d, *with respective cosine sequences* $\sigma_0, \sigma_1, \ldots, \sigma_d$ *and* $\rho_0, \rho_1, \ldots, \rho_d$. *Then for* $1 \leq i \leq d-1$

$$k = \frac{(\sigma - \sigma_2)(1-\rho) - (\rho - \rho_2)(1-\sigma)}{(\rho - \rho_2)(1-\sigma)\sigma - (\sigma - \sigma_2)(1-\rho)\rho},$$

$$b_i = k\frac{(\sigma_{i-1} - \sigma_i)(1-\rho)\rho_i - (\rho_{i-1} - \rho_i)(1-\sigma)\sigma_i}{(\rho_i - \rho_{i+1})(\sigma_{i-1} - \sigma_i) - (\sigma_i - \sigma_{i+1})(\rho_{i-1} - \rho_i)}$$

$$c_i = k\frac{(\sigma_i - \sigma_{i+1})(1-\rho)\rho_i - (\rho_i - \rho_{i+1})(1-\sigma)\sigma_i}{(\rho_i - \rho_{i+1})(\sigma_{i-1} - \sigma_i) - (\sigma_i - \sigma_{i+1})\rho_{i-1} - \rho_i)}$$

$$c_d = k\sigma_d \frac{\sigma - 1}{\sigma_{d-1} - \sigma_d} = k\rho_d \frac{\rho - 1}{\rho_{d-1} - \rho_d},$$

and the denominators are never zero.

Let Γ be a distance-regular graph with diameter $d \geq 3$. Then for any complex numbers $\theta, \sigma_0, \ldots, \sigma_d$, TFAE.

(i) θ is an eigenvalue of Γ, and $\sigma_0, \sigma_1, \ldots, \sigma_d$ is the associated cosine sequence.

(ii) $\sigma_0 = 1$, and for $0 \leq i \leq d$,

$$c_i \sigma_{i-1} + a_i \sigma_i + b_i \sigma_{i+1} = \theta \sigma_i$$

where σ_{-1} and σ_{d+1} are indeterminates.

(iii) $\sigma_0 = 1$, $k\sigma = \theta$, and for $1 \leq i \leq d$,

$$c_i(\sigma_{i-1} - \sigma_i) - b_i(\sigma_i - \sigma_{i+1}) = k(\sigma - 1)\sigma_i$$

where σ_{d+1} is an indeterminate.

Characterization of tight graphs

Theorem [JKT'00]. Γ *nonbipartite dr, diam.* $d \geq 3$*, and eigenvalues* $\theta_0 > \theta_1 > \cdots > \theta_d$*.* *Let* $\theta = \theta_1$ *and* $\theta' = \theta_d$ *with respective cosine sequences* $\sigma_0, \sigma_1, \ldots, \sigma_d$ *and* $\rho_0, \rho_1, \ldots, \rho_d$*.* *Let* $\varepsilon = (\sigma\rho - 1)/(\rho - \sigma) > 1$*. TFAE*

(i) Γ *is tight.*

(ii) $\dfrac{\sigma\sigma_{i-1} - \sigma_i}{(1+\sigma)(\sigma_{i-1} - \sigma_i)} = \dfrac{\rho\rho_{i-1} - \rho_i}{(1+\rho)(\rho_{i-1} - \rho_i)}$

$(1 \leq i \leq d)$ *and the denominators are nonzero.*

(iii) $\sigma_i\rho_i - \sigma_{i-1}\rho_{i-1} = \varepsilon(\sigma_{i-1}\rho_i - \rho_{i-1}\sigma_i)$ $(1 \leq i \leq d)$*.*

Parametrization

Theorem [JKT'00]. *Γ nonbip., dr, diam. $d \geq 3$, and let $\sigma_0, \sigma_1, \ldots \sigma_d, \varepsilon, h \in \mathbb{C}$ be scalars. TFAE*

(i) *Γ is tight, $\sigma_0, \sigma_1, \ldots \sigma_d$ is the cosine sequence corresponding to θ_1, associated parameter* $\varepsilon = (k^2 - \theta_1\theta_d) \,/\, (k(\theta_1 - \theta_d))$ and

$$h \;=\; (1-\sigma)(1-\sigma_2) \,/\, ((\sigma^2 - \sigma_2)(1-\varepsilon\sigma)).$$

(ii) $\sigma_0 = 1$, $\sigma_{d-1} = \sigma\sigma_d$, $\varepsilon > -1$, $k = h(\sigma-\varepsilon)/(\sigma-1)$, $c_d = k$, *for* $1 \leq i \leq d-1$

$$b_i, c_i == h\frac{(\sigma_{i\mp1} - \sigma_1\sigma_i)(\sigma_{i\pm1} - \varepsilon\sigma_i)}{(\sigma_{i\mp1} - \sigma_{i\pm1})(\sigma_{i\pm1} - \sigma_i)}$$

and denominators are all nonzero.

(iii) Γ *is nonbipartite and* $E_\theta \circ E_{\theta'}$ *is a scalar multiple of a primitive idempotent* E_τ.

(iv) Γ *is nonbipartite and for a vertex* x *the irreducible* $T(x)$*-module with endpoint* 1 *is short.*

Moreover, if Γ *is tight, then the above conditions are satisfied for all edges and vertices of* Γ,

$$\{\theta, \theta'\} = \{\theta_1, \theta_d\}$$

and $\tau = \theta_{d-1}$.

Theorem. Γ *antipodal distance-regular, diam. 4, eigenvalues* $k=\theta_0>\cdots>\theta_4$, $p,q\in\mathbb{N}$. *TFAE*

(i) Γ *is tight,*

(ii) the antipodal quotient is

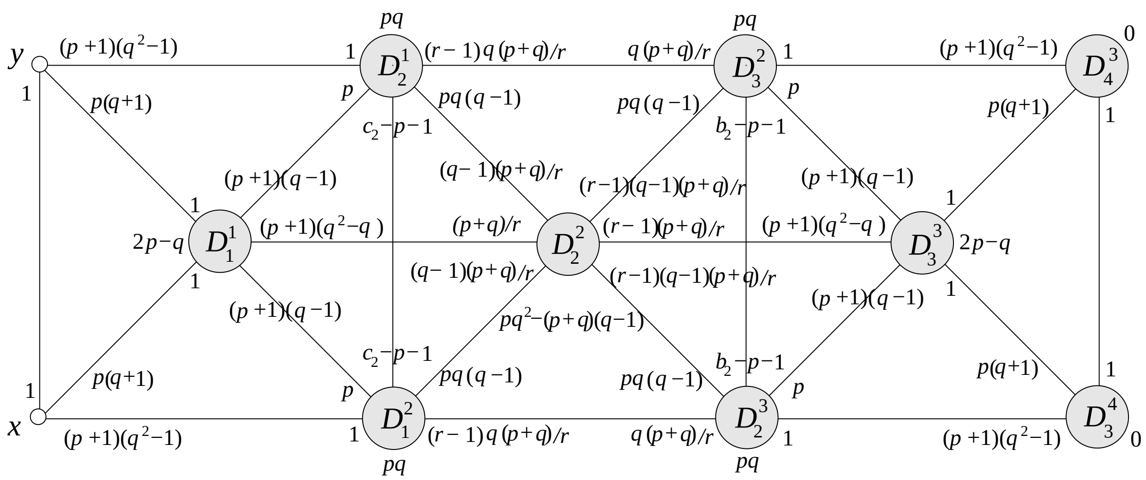
$$\mathrm{SRG}(k=q(pq+p+q),\ \lambda=p(q+1),\ \mu=q(p+q)),$$

(iii) $\theta_0=q\theta_1$, $\theta_1=pq+p+q$, $\theta_2=p$, $\theta_3=-q$, $\theta_4=-q^2$,

(iv) for each $v\in V(\Gamma)$ *the local graph of* v *is*

$$\mathrm{SRG}(k'=p(q+1),\ \lambda'=2p-q,\ \mu'=p)\ \text{(ev. } \boldsymbol{p},\boldsymbol{-q}\text{)}.$$

If Γ satisfies (i)-(iv) and $\boldsymbol{r}$ is its antipodal class size, then we call it an **antipodal tight graph AT4($\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}$).**



y
x
D_2^1
D_3^2
D_4^3
D_1^1
D_2^2
D_3^3
D_1^2
D_2^3
D_3^4
$(p+1)(q^2-1)$
$p(q+1)$
pq
$(r-1)q(p+q)/r$
$q(p+q)/r$
$pq(q-1)$
c_2-p-1
b_2-p-1
$(p+1)(q-1)$
$(q-1)(p+q)/r$
$(r-1)(q-1)(p+q)/r$
$(p+1)(q^2-q)$
$(p+q)/r$
$(r-1)(p+q)/r$
$2p-q$
$pq^2-(p+q)(q-1)$
p
1
0

Theorem. *Γ antipodal tight graph $AT4(p,q,r)$. Then*

(i) $pq(p+q)/r$ *is even,*

(ii) $r(p+1) \leq q(p+q)$,
with equality iff μ-graphs are complete,

(iii) $r \mid p+q$,

(iv) $p \geq q-2$, *with equality iff* $q^4_{44} = 0$.

(v) $p+q \mid q^2(q^2-1)$,

(vi) $p+q^2 \mid q^2(q^2-1)(q^2+q-1)(q-2)$.

Ruled out cases

#	k	b^+	$-b^-$	r	(I)	(II)	(III)
1	45	3	3	2	*		
2	45	3	3	6	*	*	
3	56	2	4	4			*
4	56	2	4	8			*
5	81	6	3	9		*	
6	96	4	4	8		*	
7	115	3	5	5			*
8	115	3	5	8	*		
9	115	3	5	10			*
10	117	9	3	4	*	*	
11	117	9	3	6		*	
12	117	9	3	9		*	*
13	175	5	5	2	*		
14	176	8	4	6		*	
15	189	15	3	6	*	*	
16	189	15	3	2	*		
17	204	4	6	3			*
18	204	4	6	4			*
19	261	21	3	4		*	
20	414	9	6	2	*		*

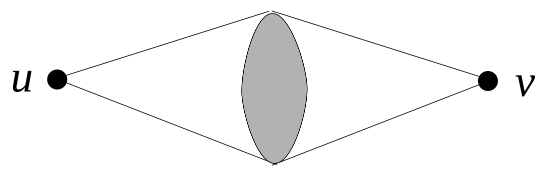
Open cases

#	k	p	q	r	μ	μ-graph
1	81	6	3	3	9	$\nexists$ by [JK]
3	96	4	4	4	8	$\nexists$ by [JK]
7	175	5	5	5	10	$\nexists$ by [JK]
9	189	15	3	3	18	$\nexists$ by [JK]
2	96	4	4	2	16	$2 \cdot K_{4,4}$
4	115	3	5	2	20	$2 \cdot$ Petersen
5	115	3	5	4	10	Petersen
6	117	9	3	2	18	$K_{9,9}$???
8	176	8	4	3	16	$2 \cdot K_{8,8}$
10	204	4	6	2	30	$5 \cdot K_{3\times 2}$
11	204	4	6	5	12	$2 \cdot K_{3\times 2}$
12	261	21	3	2	36	no idea???
13	288	6	6	2	36	$3 \cdot K_{6,6}$
14	288	6	6	3	24	$2 \cdot K_{6,6}$
15	329	5	7	2	42	$7 \cdot K_6$
16	336	16	4	2	40	$2 \cdot K_{5\times 4}$
17	416	20	4	2	48	$2 \cdot K_{6\times 4}$

Known examples of AT4 family

#	graph	k	p	q	r	μ	μ-graph
1	! Conway-Smith	10	1	2	3	2	K_2
2	! $J(8,4)$	16	2	2	2	4	$K_{2,2}$
3	! halved Q_8	28	4	2	2	6	$K_{3\times 2}$
5	$3.O_6^-(3)$	45	3	3	3	6	$K_{3,3}$
4	! Soicher1	56	2	4	3	8	$2\cdot K_{2,2}$
6	$3.O_7(3)$	117	9	3	3	12	$K_{4\times 3}$
7	Meixner1	176	8	4	2	24	$2\cdot K_{3\times 4}$
8	Meixner2	176	8	4	4	12	$K_{3\times 4}$
9	Soicher2	416	20	4	3	32	$\overline{K_2}$-ext. of Q_5
10	$3.Fi_{24}^-$	31671	351	9	3	1080	$O_8^+(3)$

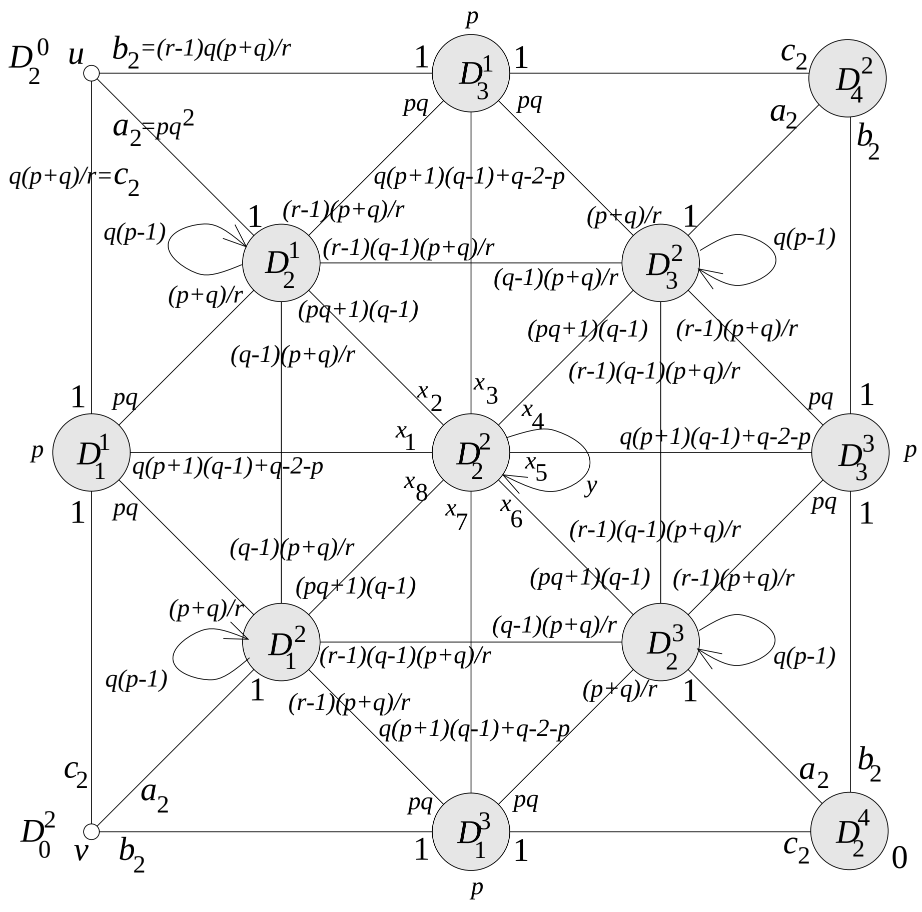
Γ graph, diam.$d \geq 2$, $u, v \in V(\Gamma)$, dist$(u,v) = 2$. The ***μ-graph*** of u and v is the graph induced by $D_1^1(u,v) = \Gamma(u) \cap \Gamma(v)$.



Lemma [JK'03]. Γ *distance-regular, local graphs are strongly regular* (v', k', λ', μ'). *Then*

1. *μ-graphs of Γ are μ'-regular,*
2. *$c_2\mu'$ is even, and*
3. *$c_2 \geq \mu' + 1$, (equality $\Longleftrightarrow$ μ-graphs are K_μ),*

For the AT4 family we know also $r \,|\, p{+}q$, $p \geq q{-}2$.



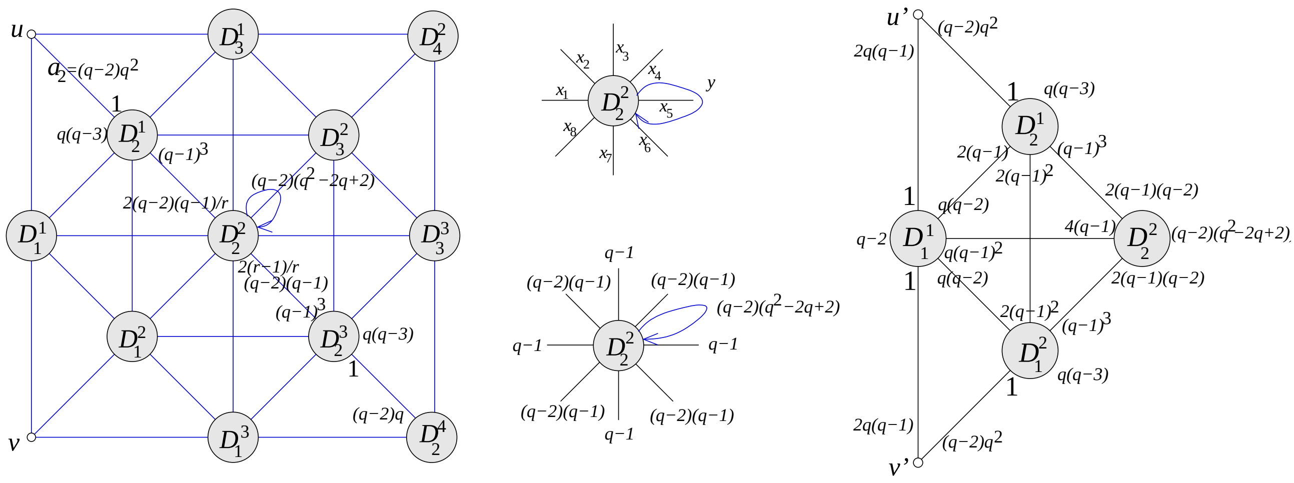
p
D_2^0 u b_2=(r-1)q(p+q)/r
1 D_3^1 1
c_2 D_4^2
pq pq
a_2=pq^2
a_2
b_2
q(p+q)/r=c_2
q(p+1)(q-1)+q-2-p
1 (r-1)(p+q)/r
(p+q)/r 1
q(p-1)
q(p-1)
D_2^1 (r-1)(q-1)(p+q)/r
(q-1)(p+q)/r D_3^2
(p+q)/r
(pq+1)(q-1)
(pq+1)(q-1)
(r-1)(p+q)/r
(q-1)(p+q)/r
(r-1)(q-1)(p+q)/r
1 pq
x_2 x_3
pq 1
x_4
p D_1^1
x_1 D_2^2
q(p+1)(q-1)+q-2-p
D_3^3 p
q(p+1)(q-1)+q-2-p
x_5 y
x_8 x_7 x_6
1 pq
pq 1
(q-1)(p+q)/r
(r-1)(q-1)(p+q)/r
(pq+1)(q-1)
(pq+1)(q-1)
(r-1)(p+q)/r
(p+q)/r
(q-1)(p+q)/r
D_1^2 (r-1)(q-1)(p+q)/r
D_2^3
q(p-1)
q(p-1)
1 (p+q)/r 1
(r-1)(p+q)/r
q(p+1)(q-1)+q-2-p
c_2 a_2
a_2 b_2
pq pq
D_0^2 v b_2
1 D_1^3 1
c_2 D_2^4 0
p

The case $p = q - 2$

Theorem [J'02]. *Let* Γ *be* $\mathrm{AT4}(p, q, r)$.
Let $p = q - 2$, *i.e.*, $q_{44}^{4} = 0$. *Then* $\forall v \in V(\Gamma)$
$\Gamma_2(v)$ *induces an antipodal drg with diam.* 4.
If $r = 2$ *then* Γ *is* 2*-homogeneous.*

Example: The Soicher1 graph ($q = 4$ and $r = 3$).
$\Gamma_2(v)$ induces $\{32, 27, 8, 1; 1, 4, 27, 32\}$
(Soicher has found this with the aid of a computer).
The antipodal quotient of this graph is the strongly regular graph, and it is the second subconstituent graph of the second subconstituent graph of the McLaughlin graph.

All local graphs are the incidence graphs of $AG(2, 4)\backslash$ a parallel class ($\{4, 3, 3, 1; 1, 1, 3, 4\}$), i.e., the antipodal 4-covers of $K_{4,4}$.



u
v
a2=(q−2)q2
1
q(q−3)
(q−1)3
2(q−2)(q−1)/r
(q−2)(q2−2q+2)
2(r−1)/r
(q−2)(q−1)
(q−1)3
q(q−3)
1
(q−2)q
x1
x2
x3
x4
x5
x6
x7
x8
y
q−1
(q−2)(q−1)
(q−2)(q2−2q+2)
u'
v'
(q−2)q2
2q(q−1)
q−2
4(q−1)
2(q−1)
2(q−1)2
q(q−2)
q(q−1)2
2(q−1)(q−2)

Proof. Cauchy-Schwartz inequality

$$||w_{11}||^2||E\hat{u}+E\hat{v}||^2 - w_{11}(E\hat{u}+E\hat{v}) \geq 0,$$

where

$$w_{11} = \sum_{w \in D_1^1(u,v)} E\hat{w}$$

for $\partial(u,v)=2$, simplifies to

$$\boxed{1-\gamma_2+\tfrac{c_2(\gamma_2+\gamma_2^2-2\gamma_1^2)}{1+\gamma_2}+\mu'(\gamma_1-\gamma_2)\geq 0.}$$

where $\{\gamma_i\}$ is the cosine seq. corr. to E, and μ' is the valency of $D_1^1(u,v)$.

Equality:

$$w_{11}(u,v) = c(E\hat{u} + E\hat{v}),$$

where c is a constant, for any u, v, $\partial(u,v) = 2 \iff$
E corr. to θ_4 and $p = q - 2$ $(c = -2(q-1)/4)$, or
E corr. to θ_1 and $r = 2$ $(c = (p+q)/2)$.

The case $q \mid p$

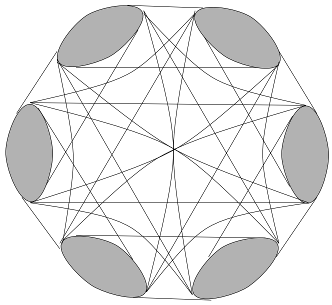
Lemma [JK'02]. *Let* Γ *be a* AT4(p,q,r). *Then* Γ *is pseudogeometric* $(p+1+p/q, q, p/q)$ *iff* $q \mid p$.

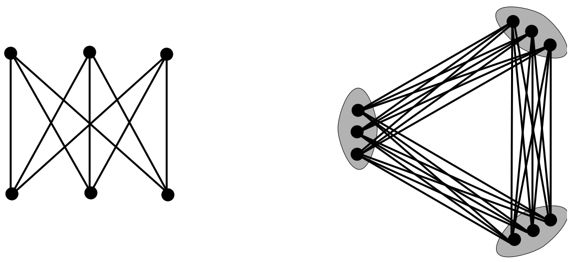
Conjecture [J'03].

AT4(p,q,r) *family is* **finite** *and either*

1. $(p,q,r) \in \{(1,2,3),(20,4,3),(351,9,3)\}$,
2. $q \,|\, p$ *and* $r = q$ *or* $r = 2$, *i.e.,* $\mathbf{AT4(}\boldsymbol{qs,q,q}\mathbf{)}$ *or* $\mathbf{AT4(}\boldsymbol{qs,q,}\mathbf{2)}$ *(a local graph is pseudogeometric),*
3. $p = q-2$ *and* $r = q$ *or* $r = 2$, *i.e.,* $\mathbf{AT4(}\boldsymbol{q}\mathbf{-2,}\,\boldsymbol{q,q}\mathbf{)}$ *or* $\mathbf{AT4(}\boldsymbol{q}\mathbf{-2,}\,\boldsymbol{q,}\mathbf{2)}$. *($\tilde{\Gamma}_2(x)$ is strongly regular),*

Complete multipartite graphs:



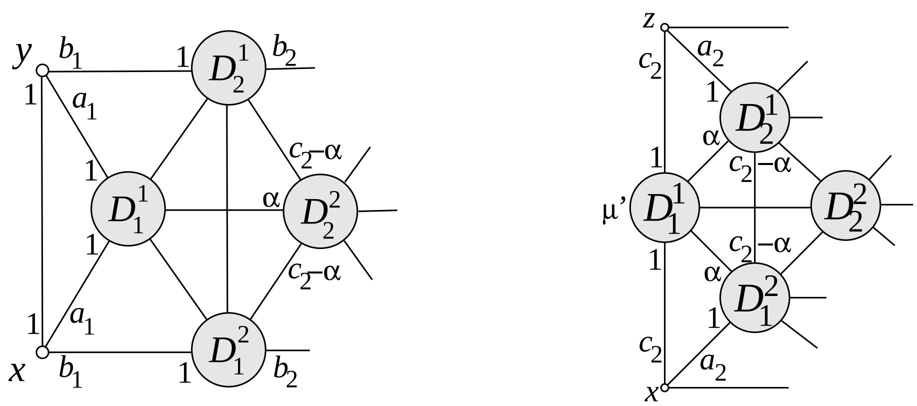


$K_{t\times n}$, and examples $K_{2\times 3} = K_{3,3}$ and $K_{3\times 3}$.

$\mathbf{CAB_2}$ property and parameter $\boldsymbol{\alpha}$

Γ drg, $d \geq 2$, $a_2 \neq 0$, $\partial(x,z) = 2 = \partial(y,z)$, $\partial(x,y) = 1$:

$$\alpha := \Gamma(z) \cap \Gamma(y) \cap \Gamma(x).$$



We say $\exists$ α when $\alpha = \alpha(x,y,z)$ $\quad \forall (x,y,z) \in (V\Gamma)^3$ s.t. $\partial(x,z) = 2 = \partial(y,z)$, $\partial(x,y) = 1$.

$K_{t\times n} := \overline{t\cdot K_n}$ $(= K_n^t)$, for example $K_{2\times 3} = K_{3,3}$.

Γ k-regular, v vertices and let any two vertices at distance 2 have $\mu = \mu(\Gamma)$ common neighbours. Then it is called **co-edge-regular** with parameters (v, k, μ).

Lemma [JK'03]. *Γ distance-regular, diam. $d \geq 2$, $K_{t\times n}$ as μ-graphs, $a_2 \neq 0$ and $\exists\ \alpha \neq 1$. Then*

(i) $c_2 = nt$, each local graph of Γ is co-edge-regular with parameters $(v' = k, k' = a_1, \mu' = n(t-1))$ and $\alpha a_2 = c_2(a_1 - \mu')$,

(ii) $\alpha = t$ or $\alpha = t - 1$.

When do we know the μ-graphs?

Theorem [JK]. Γ *distance-regular, diam.* $d \geq 2$, $a_2 \neq 0$, *locally* $\mathrm{SRG}(v', k', \lambda', \mu')$, *and* $\exists\, \alpha \geq 1$. *Then*

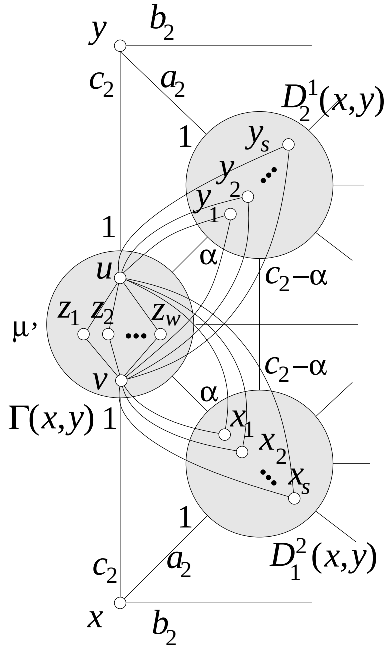
(i) If $c_2 > \mu' + 1$ *and* $2c_2 < 3\mu' + 6 - \alpha$, *then* **the μ-graphs are $K_{t\times n}$**, $n = c_2 - \mu'$, $t = c_2/n$.

(ii) If $\alpha = 1$ *and* $\mu' \neq 0$, *then* $c_2 = 2\mu'$, $\lambda' = 0$ *and* **the μ-graphs are $K_{\mu',\mu'}$**,

(iii) If $\alpha = 2$, $2 \leq \mu'$ *and* $c_2 \leq 2\mu'$, *then* $c_2 = 2\mu'$ *and the* **μ-graphs** *are* $\boldsymbol{K_{\mu',\mu'}}$ *or* $\boldsymbol{K_{3\times\mu'}}$.

Let $x, y \in \Gamma$, s.t. $\partial(x, y) = 2$ and let $M = D_1^1(x, y)$.
Then M induces the graph of valency μ' on c_2 vertices.
Let $u, v \in M$, s.t. $\partial(u, v) = 2$

Suppose $D_1^1(u, v) \cap M$
$= \{z_1, z_2, \ldots, z_t\}$
where $t < p$.



Then $D_1^1(u, v) \cap D_2^1(x, y)$
$= \{y_1, y_2, \ldots, y_s\}$

and $D_1^1(u, v) \cap D_1^1(x, y)$
$= \{x_1, x_2, \ldots, x_s\}$

where $\mu' = s + t$.

Corollary [JK]. *The μ-graphs of* AT4(qs, q, q) *are* $K_{(s+1)\times q}$ *and* $\alpha = s+1$.

The μ-graphs of the Patterson graph (and of any other graph P with the same intersection array) are $K_{4,4}$ and $\alpha = 2$.

Theorem [JK]. Γ *drg,* $K_{\mu',\mu'}$ *as* μ*-graphs* and $\alpha = 2$. *Then the local graphs of* Γ *are* $\mathrm{GQ}(*, \mu' - 1)$ *with regular points. A line size* c *satisfies* $v' = c(k' - c + 2)$.
If the local graphs of Γ *are* $\mathrm{SRG}(v', k', \lambda', \mu')$*, then they are* $\mathbf{GQ(\lambda' + 1, \mu' - 1)}$.
In particular, a local graph of P *is the point graph of the* **unique** *generalized quadrangle* $\mathrm{GQ}(3, 9)$ *with all points regular.*

We will use uniqueness of small **generalized quadrangles** with all points regular to prove uniqueness of much larger object.

Lemma. *Let Γ have a CAB_2 property with μ-graphs $K_{t\times n}$, $n \geq 2$, $t \geq 3$ and let $\alpha \geq 3$. Let xyz be a triangle of Γ and L be a lower bound on the valency of $\Delta(x,y,z)$. Then*

$$(v'' - 1 - k'')\mu'' \leq k''(k'' - 1 - L),$$

with equality iff $\forall$ edges xy $\Delta(x,y)$ is $\mathrm{SRG}(v'', k'', \lambda'', \mu'')$*, where $\lambda'' = L$.*

We derive the following lower bound:

$$L := \alpha - 2 + (n-1)\Big((t-3)n - (\alpha - 3)\Big).$$

Classification of the AT4(qs, q, q) family

Theorem [JK]. *The μ-graphs of $\Gamma = \mathrm{AT4}(p, q, r)$ are* **complete multipartite** *graphs $K_{t \times n}$ iff Γ is*

1. the Conway-Smith graph (locally Petersen graph),
2. the Johnson graph $J(8, 4)$ (locally GQ$(3, 1)$),
3. the halved 8-cube (locally $T(8)$),
4. the $3.O_6^-(3)$ graph (locally GQ$(4, 2)$),
5. the Meixner2 graph (locally locally GQ$(3, 3)$),
6. the $3.O_7(3)$ graph (locally locally locally GQ$(2, 2)$).

The $3.O_6^-(3)$ graph

is a **3-cover** of the graph Γ,
defined on 126 points of one kind in $PG(5,3)$,
provided with a quadratic form
of a non-maximal Witt index
and two points **adjacent** when they are orthogonal.

It can be described with Hermitean form in PG(3,4).
It has **378** vertices and valency **45**.

Then **the local graphs of** Γ and its covers are $\mathbf{GQ(4,2)}$.

The $3.O_7(3)$ graph

is a **3-cover** of the graph Γ,
defined on the hyperbolic points in $PG(6,3)$,
provided with a nondegenerate quadric,
and points **adjacent** when they are orthogonal.

It can be described in terms of a system of complex vectors found in $\mathbb{A}$TLAS (p.108).
It has **1134** vertices and valency **117**.

Then **(the local graphs of)**^{3} Γ and its covers are **GQ(2, 2)**.

The Meixner2 graph

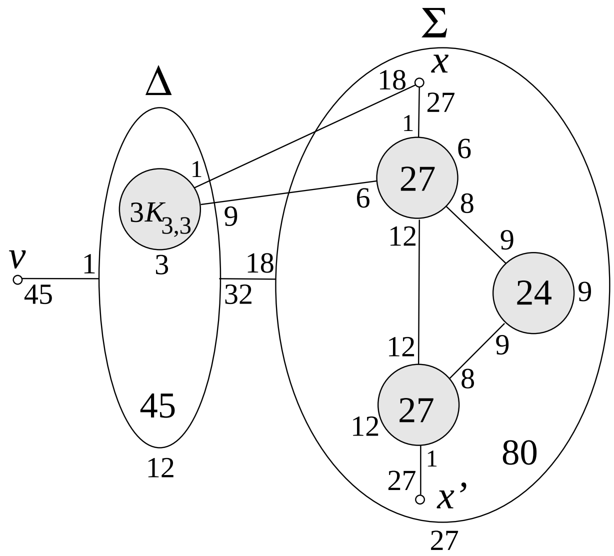
The graph $\boldsymbol{\mathcal{U}_n}$ has for its **vertices** the nonisotropic points of the n-dim. vector space over $GF(4)$ with a nondegenerate Hermitean form,
and two points **adjacent** if they are orthogonal.

$\mathcal{U}_4$ is GQ$(3,3)$ (W_3), and $\mathcal{U}_{n+1}$ is locally $\mathcal{U}_n$.

The Meixner2 graph is $\mathcal{U}_6$, so it has **2688** vertices, valency **176** and **(the local graphs of)**^{2} it are $\mathbf{GQ(3,3)}$.

The $3.O_6^-(3)$ graph

$\{45, 32, 12, 1; 1, 6, 32, 45\}$, distance-transitive,
3-cover of $SRG(126, 45, 12, 18)$, not Q-poly.,
locally **generalized quadrangle** $GQ(4, 2)$.



The $3.O_7^-(3)$ graph

We obtained that the antipodal quotient of Γ
has parameters $\{117, 80; 1, 36\}$,
with $\lambda = 36$ and μ-graphs $3{\cdot}K_{4\times 3}$,
whose local graphs have parameters $\{36, 20; 1, 9\}$,
with $\lambda' = 15$ and μ-graphs $K_{3\times 3}$,
whose local graphs have parameters $\{15, 8; 1, 6\}$,
with $\lambda'' = 6$ and μ-graphs $K_{2\times 3}$,
whose local graphs have parameters $\{6, 4; 1, 3\}$,
with $\lambda''' = 1$ and μ-graphs $3{\cdot}K_1$.

Meixner2

We obtained that the antipodal quotient of Meixner2 has parameters $\{176, 135; 1, 48\}$,
with $\lambda = 40$ and μ-graphs $4 \cdot K_{3\times 4}$,
whose local graphs have parameters $\{40, 27; 1, 8\}$,
with $\lambda' = 12$ and μ-graphs $K_{4,4}$,
whose local graphs have parameters $\{12, 9; 1, 4\}$,
with $\lambda'' = 2$ and μ-graphs $4 \cdot K_1$.

The Patterson graph

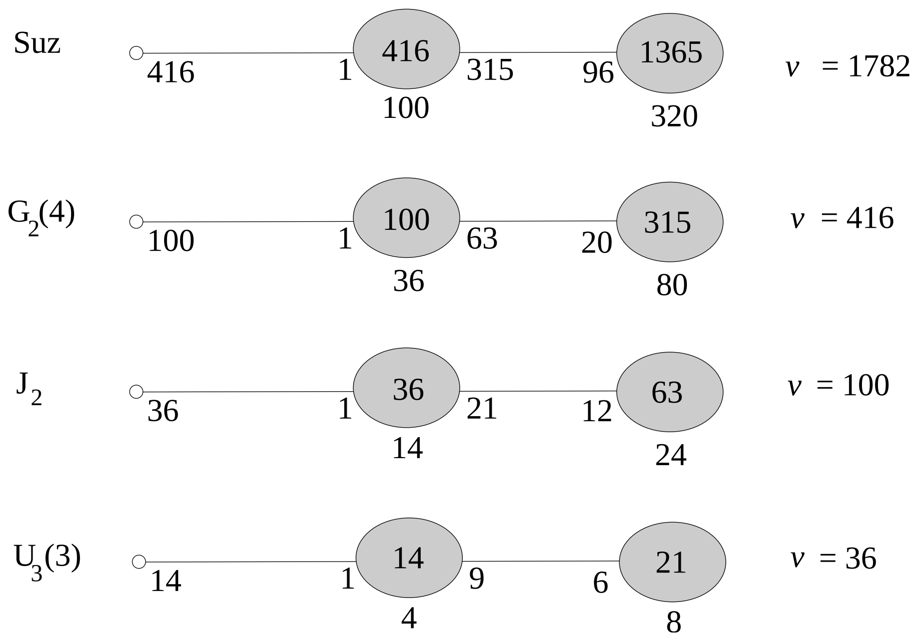
is defined as the **graph** Γ with:

22.880 centers (of order 3) of the Sylow 3-groups of the sporadic simple group of Suzuki (**Suz**, see ATLAS) of order $2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ as the **vertices**,

two **adjacent** iff they generate an abelian subgroup of order 3^2.

Problem ([BCN,p.410]): **Is this graph unique?**
(uniquely determined by its regularity properties)

The Suzuki tower

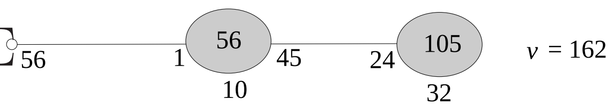


The derived design of the Steiner system $S(4,7,23)$ defines the *McLaughlin graph*, i.e., the *unique* SRG$(275,112,30,56)$.

This graph is locally **GQ(3, 9)** and the second subconstituent graph is a unique SRG$(162,56,10,24)$.

We can find it in the Suz as an induced subgraph.

$\Gamma_2(\text{McLaughlin}) \;=\; \Sigma$



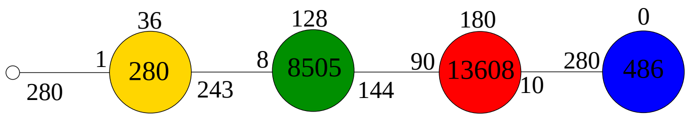
An alternative definition of the Patterson graph:
Induced Σ's in Suz, adjacent when disjoint

11-cliques: partitions of Suz in 11 Σ's

The Patterson graph is **distance-regular** with intersection array

$$\{280, 243, 144, 10; 1, 8, 90, 280\}$$

and eigenvalues $280^1,\ 80^{364},\ 20^{5940},\ -8^{15795},\ -28^{780}.$



group $\quad$ Suz.2 $\quad$ (distance-transitive)

point stabilizer $\quad 3 \cdot U_4(3).(2^2)_{133}$

locally $\quad$ GQ$(9, 3)$ $\quad$ (group $U_4(3).D_8$).

Theorem [BJK]. *A distance-regular graph* ***P*** *with intersection array*

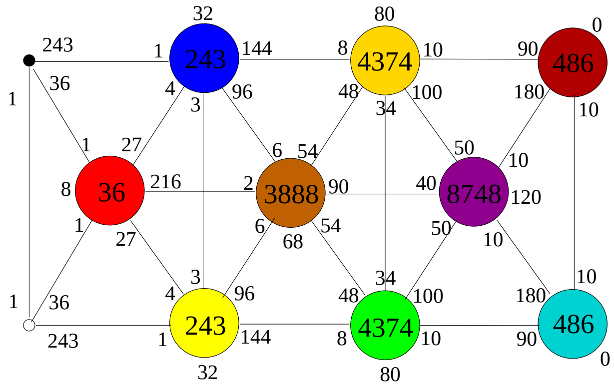
$$\{280, 243, 144, 10; 1, 8, 90, 280\} \qquad (22.880 \textit{ vertices})$$

is **unique**.

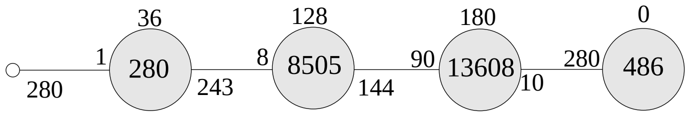
For example, the **icosahedron** is a unique graph, that is **locally pentagon**.

The **Petersen graph** is a unique strongly-regular graph $\mathbf{(10, 3, 0, 1)}$, i.e., $\{3, 2; 1, 1\}$.

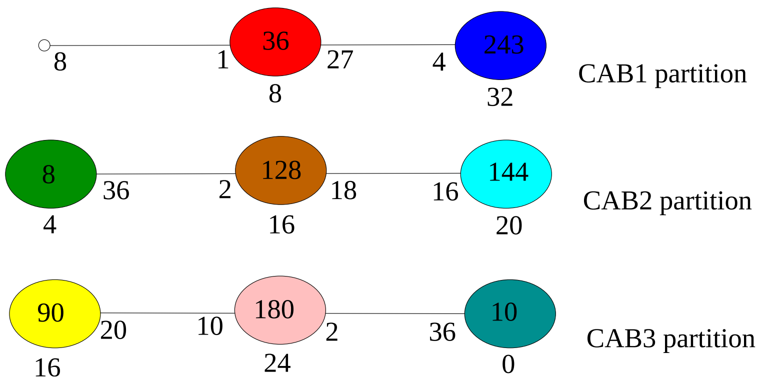
The distance-partitions of Γ *corresp. to an edge* (i.e., the collection of nonempty sets $D_j^h(x,y)$) are also *equitable* ($\forall xy \in E\Gamma$):



So the Patterson graph is **1-homogeneous**.



The 1-homogeneous property and the CAB property.



Corollary. *If an AT4(p, q, r) has a μ-graph that is not complete multipartite, then either*

1. $$\frac{(p+q)(2q+1)}{3(p+2)} \geq r \geq q+1,$$
2. $r = q-1$ *if and only if* $p = q-2$
3. $r \leq q-2.$

Γ tight, diam. 4, $\alpha = 2$, $K_{t+1,t+1}$ as μ-graphs.

Example 1: If local graphs are $\mathbf{GQ}(t^2, t)$, then

$$\{(t^2+1)(t^3+1), t^5, t^2(t+1)(t-1)^2, (t-1)(t^2-t-1);$$
$$1, 2(t+1), 2t^2(t+2), (t^2+1)(t^3+1)\}.$$

For $t = 2$ we get the $3.O_6^-(3)$ graph and
for $t = 3$ the Patterson graph,
for $t = 4$ the existence is **OPEN**,

In Ex.1, the case $t = 4$ we have the following feasible intersection array

$$\{1105, 1024, 720, 33; 1, 10, 192, 1105\},$$

and eigenvalues:

$$1105^1,\ 255^{1911}, 55^{116688}, -15^{424320},\ -65^{8330}.$$

If it exist, then it has 551,250 vertices
($k_2 = 113,152$, $k_3 = 424,320$, $k_4 = 12,672$) and its local graphs are $GQ(16,4)$ with all points being regular this is most probably the hermitian generalized quadrangle $H(3,16)$).

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