IV. Geometry

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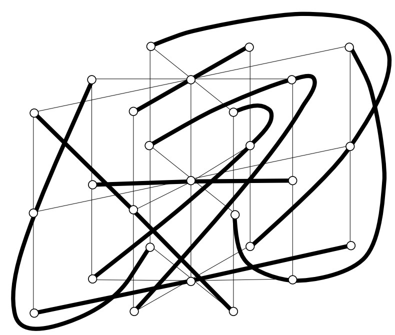


A unique spread

in GQ(3,3)=W(3)

A triple (P, L, I), i.e., (points,lines,incidence), is called a **partial geometry** $pg(R, K, T)$, when $\forall \ell, \ell' \in L$, $\forall p, p' \in P$:

- $|\ell| = K$, $|\ell \cap \ell'| \leq 1$,
- $|p| = R$, at most one line on p and p',
- if $p \notin \ell$, then there are exactly T points on ℓ that are collinear with p.

The **dual** (L, P, I^t) of a $pg(R, K, T)$ is again a partial geometry, with parameters (K, R, T).

Classification

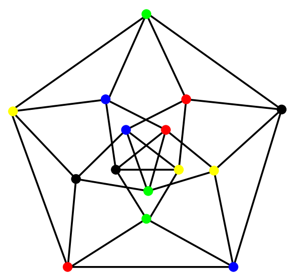
We divide partial geometries into four classes:

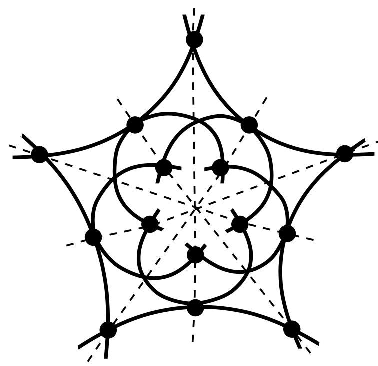
1. $T = K$: 2-$(v, K, 1)$ design,
2. $T = R - 1$: net,

   $T = K - 1$: transversal design,
3. $T = 1$: a generalized quadrangle $\mathrm{GQ}(K-1, R-1)$,
4. For $1 < T < \min\{K - 1, R - 1\}$ we say we have a **proper partial geometry**.

A $pg(t + 1, s + 1, 1)$ is a **generalized quadrangle** $GQ(s, t)$.

An example





L(Peterson) is the point graph of the GQ$(2,2)$ minus a spread (where spread consists of antipodal classes).

What about trivial examples?

Pseudo-geometric

The **point graph** of a $pg(P, L, I)$ is the graph with vertex set $X = P$ whose edges are the pairs of collinear points (also known as the *collinearity graph*).

The point graph of a $pg(R, K, T)$ is SRG:
$k = R(K-1)$, $\lambda = (R-1)(T-1)+K-2$, $\mu = RT$, and eigenvalues $r = K - 1 - T$ and $s = -R$.

A SRG is called **pseudo-geometric** (R, K, T) if its parameters are as above.

Quadratic forms

A **quadratic form $\boldsymbol{Q(x_0, x_1, \ldots, x_n)}$** over GF$(q)$ is a homogeneous polynomial of degree 2,

i.e., for $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ and an $(n+1)$-dim square matrix C over GF(q):

$$Q(\boldsymbol{x}) = \sum_{i,j=0}^{n} c_{ij} x_i x_j = \boldsymbol{x} C \boldsymbol{x}^T.$$

A **quadric** in PG(n, q) is the set of **isotropic** points:

$$Q = \{\langle \boldsymbol{x} \rangle \mid Q(\boldsymbol{x}) = 0\},$$

where $\langle \boldsymbol{x} \rangle$ is the 1-dim. subspace of $GF(q)^{n+1}$ generated by $\boldsymbol{x} \in (\mathrm{GF}(q))^{n+1}$.

Two quadratic forms $Q_1(\boldsymbol{x})$ and $Q_2(\boldsymbol{x})$ are **projectively equivalent** if there is an invertible matrix A and $\boldsymbol{\lambda} \neq 0$ such that

$$Q_2(\boldsymbol{x}) = \boldsymbol{\lambda} Q_1(\boldsymbol{x}A).$$

The **rank** of a quadratic form is the smallest number of indeterminates that occur in a projectively equivalent quadratic form.

A quadratic form $Q(x_0, \dots, x_n)$ (or the quadric Q in $\mathrm{PG}(n, q)$ determined by it) is **nondegenerate** if its rank is $n+1$. (i.e., $Q \cap Q^{\perp} = 0$ and also to $Q^{\perp} = 0$).

For q odd a subspace U is **degenerate** whenever

$$U \cap U^{\perp} \neq \emptyset,$$

i.e., whenever its orthogonal complement $U^{\perp}$ is degenerate, where $\perp$ denotes the inner product on the vector space $V(n+1, q)$ defined by

$$(x, y) := Q(x+y) - Q(x) - Q(y).$$

Isotropic spaces

A **flat** of projective space $\mathrm{PG}(n, q)$
(defined over $(n+1)$-dim. space V)
consists of 1-dim. subspaces of V
that are contained in some subspace of V.

A flat is said to be **isotropic** when all its points are isotropic.

The dimension of maximal isotropic flats will be determined soon.

Theorem. *A nondegenerate quadric* $Q(\boldsymbol{x})$ *in* $\mathrm{PG}(n,q)$*,* q *odd, has the following canonical form*

(i) *for* n *even:* $Q(\boldsymbol{x}) = \sum_{i=0}^{n} x_i^2$*,*

(ii) *for* n *odd:*

(a) $Q(\boldsymbol{x}) = \sum_{i=0}^{n} x_i^2$*,*

(b) $Q(\boldsymbol{x}) = \eta x_0^2 + \sum_{i=1}^{n} x_i^2$*, where* η *is not a square.*

Theorem. *Any nondegenerate quadratic form $Q(\boldsymbol{x})$ over $\mathrm{GF}(q)$ is projectively equivalent to*

(i) *for* $n=2s$: $\mathcal{P}_{2s} = x_0^2+\sum_{i=1}^{s} x_{2i}x_{2i-1}$ (**parabolic**),

(ii) *for* $n = 2s-1$

(a) $\mathcal{H}_{2s-1} = \sum_{i=0}^{s-1} x_{2i}x_{2i+1}$ (**hyperbolic**),

(b) $\mathcal{H}_{2s-1} = \sum_{i=1}^{s-1} x_{2i}x_{2i+1} + f(x_0, x_1)$, (**elliptic**)

where f is an irreducible quadratic form.

The dimension of maximal isotropic flats:

Theorem. *A nondegenerate quadric* Q *in* $\mathrm{PG}(n,q)$ *has the following number of points and maximal projective dim. of a flat* F, $F \subseteq Q$*:*

(i) $\dfrac{q^n - 1}{q-1}$, $\dfrac{n-2}{2}$, *parabolic*

(ii) $\dfrac{(q^{(n+1)/2} - 1)(q^{(n+1)/2} + 1)}{q-1}$, $\dfrac{n-1}{2}$ *hyperbolic,*

(iii) $\dfrac{(q^{(n+1)/2} - 1)(q^{(n+1)/2} + 1)}{q-1}$, $\dfrac{n-3}{2}$ *elliptic.*