Classification

We classify strongly regular graphs into two types:

Type I (or conference) graphs: for these graphs $(n-1)(\mu-\lambda) = 2k$, which implies $\lambda = \mu-1$, $k = 2\mu$ and $n = 4\mu+1$, i.e., the strongly regular graphs with the same parameters as their complements.

They exist iff n is the sum of two squares.

Type II graphs: for these graphs $(\mu-\lambda)^2+4(k-\mu)$ is a perfect square Δ^2, where Δ divides $(n-1)(\mu-\lambda)-2k$ and the quotient is congruent to $n-1 \pmod 2$.

Paley graphs

q a prime power, $q \equiv 1 \pmod 4$ and set $\mathbb{F} = \mathrm{GF}(q)$. The **Paley graph** $\boldsymbol{P(q)}$= (V, E) is defined by:

$$V = \mathbb{F} \ \text{ and } \ E = \{(a,b) \in \mathbb{F} \times \mathbb{F} \,|\, (a-b) \in (\mathbb{F}^*)^2\}.$$

i.e., two vertices are adjacent if their difference is a non-zero square. $P(q)$ is **undirected**, since $-1 \in (\mathbb{F}^*)^2$.

Consider the map $x \longrightarrow x + a$, where $a \in \mathbb{F}$, and the map $x \longrightarrow xb$, where $b \in \mathbb{F}$ is a square or a nonsquare, to show $P(q)$ is **strongly regular** with

$$\text{valency } k = \frac{q-1}{2}, \quad \lambda = \frac{q-5}{4} \ \text{ and } \ \mu = \frac{q-1}{4}.$$

Seidel showed that these graphs are uniquely determined with their parameters for $q \leq 17$.

There are some results in the literature showing that Paley graphs behave in many ways like random graphs $G(n, 1/2)$.

Bollobás and Thomason proved that the Paley graphs contain all small graphs as induced subgraphs.

Krein conditions

Of the other conditions satisfied by the parameteres of SRG, the most important are the **Krein conditions**, first proved by Scott using a result of Krein from harmonic analysis:

$$(\sigma+1)(k+\sigma+2\sigma\tau) \leq (k+\sigma)(\tau+1)^2$$

and

$$(\tau+1)(k+\tau+2\sigma\tau) \leq (k+\tau)(\sigma+1)^2.$$

Some parameter sets satisfy all known necessary conditions. We will mention some of these.

If $k > s > t$ eigenvalues of a strongly regular graph, then the first inequality translates to

$$k \geq -s\,\frac{(2t+1)(t-s)-t(t+1)}{(t-s)+t(t+1)},$$

$$\lambda \geq -(s+1)t\,\frac{(t-s)-t(t+3)}{(t-s)+t(t+1)},$$

$$\mu \geq -s(t+1)\,\frac{(t-s)-t(t+1)}{(t-s)+t(t+1)},$$

A strongly regular graph with parameters (k, λ, μ) given by taking equalities above, where t and s are integers such that $t-s \geq t(t+3)$ (i.e., $\lambda \geq 0$) and $k > t > s$ is called a **Smith graph**.

A strongly regular graph with eigenvalues $k > \sigma > \tau$ is said to be of **(negative) Latin square type** when $\mu = \tau(\tau + 1)$ (resp. $\mu = \sigma(\sigma + 1)$).

The complement of a graph of (negative) Latin square type is again of (negative) Latin square type.

A graph of Latin square type is denoted by $\mathrm{L}_u(v)$, where $u = -\sigma$, $v = \tau - \sigma$ and it has the same parameters as the line graph of a $\mathrm{TD}_u(v)$.

Graphs of negative Latin square type ware introduced by Mesner, and are denoted by $\mathrm{NL}_e(f)$, where $e = \tau$, $f = \tau - \sigma$ and its parameters can be obtained from $\mathrm{L}_u(v)$ by replacing u by $-e$ and v by $-f$.

More examples of strongly regular graphs:

$L(K_v)$ is strongly regular with parameters

$$n = \binom{v}{2}, \quad k = 2(v-1), \quad \lambda = v-2, \quad \mu = 4.$$

For $v \neq 8$ this is the unique srg with these parameters.

Similarly, $L(K_{v,v}) = K_v \times K_v$ is strongly regular, with parameters

$$n = v^2, \quad k = 2(v-1), \quad \lambda = v-2, \quad \mu = 2.$$

and eigenvalues $\quad 2(v-1)^1, \quad v-2^{2(v-1)}, \quad -2^{(v-1)^2}.$

For $v \neq 4$ this is the unique srg with these parameters.

Steiner graph is the block (line) graph of a 2-$(v, s, 1)$ design with $v-1 > s(s-1)$, and it is strongly regular with parameters

$$n = \frac{\binom{v}{2}}{\binom{s}{2}}, \qquad k = s\Big(\frac{v-1}{s-1} - 1\Big),$$

$$\lambda = \frac{v-1}{s-1} - 2 + (s-1)^2, \qquad \mu = s^2.$$

and eigenvalues

$$k^1, \quad \left(\frac{v-s^2}{s-1}\right)^{v-1}, \quad -s^{n-v}.$$

When in a design $\mathcal{D}$ the block size is two, the number of edges of the point graph equals the number of blocks of the design $\mathcal{D}$. In this case the line graph of the design $\mathcal{D}$ is the line graph of the point graph of $\mathcal{D}$. This justifies the name: the line graph of a graph.

A point graph of a Steiner system is a complete graph, thus a line graph of a Steiner system $S(2, v)$ is the line graph of a complete graph K_v, also called the **triangular graph**.

(If $\mathcal{D}$ is a square design, i.e., $v - 1 = s(s-1)$, then its line graph is the complete graph K_v.)

The fact that Steiner triple system with v points exists for all $v \equiv 1$ or $3 \pmod 6$ goes back to Kirkman in 1847. More recently Wilson showed that the number $n(v)$ of Steiner triple systems on an andmissible number v of points satisfies

$$n(v) \geq \exp(v^2 \log v/6 - cv^2).$$

A Steiner triple system of order $v > 15$ can be recovered uniquely from its line graph, hence there are super-exponentially many SRG$(n, 3s, s+3, 9)$, for $n = (s+1)(2s+3)/3$ and $s \equiv 0$ or $2 \pmod 3$.

For $2 \le s \le v$ the **block graph** of a transversal design $\mathrm{TD}(s,v)$ (two blocks being adjacent iff they intersect) is strongly regular with parameters $n = v^2$,

$$k = s(v-1),\ \ \lambda = (v-2)+(s-1)(s-2),\ \ \mu = s(s-1).$$

and eigenvalues

$$s(v-1)^1,\ \ v - s^{s(v-1)},\ \ -s^{(v-1)(v-s+1)}.$$

Note that a line graph of $\mathrm{TD}(s,v)$ is a conference graph when $v = 2s-1$. For $s = 2$ we get **the lattice graph** $K_v \times K_v$.

The number of Latin squares of order k is asymptotically equal to

$$\exp(k^2 \log k - 2k^2)$$

Theorem (Neumaier). *The strongly regular graph with the smallest eigenvalue $-m$, $m \geq 2$ integral, is with finitely many exceptions, either a complete multipartite graph, a Steiner graph, or the line graph of a transversal design.*