III. Strongly regular graphs

- definition of strongly regular graphs
- characterization with adjacency matrix
- classification (type I in II)
- Payley graphs
- Krein condition and Smith graphs
- more examples (Steiner and LS graphs)
- feasibility conditions and a table
Definition

Two similar regularity conditions are:

(a) any two adjacent vertices have exactly $\lambda$ common neighbours,

(b) any two nonadjacent vertices have exactly $\mu$ common neighbours.

A regular graph is called strongly regular when it satisfies (a) and (b). Notation $\text{SRG}(n, k, \lambda, \mu)$, where $k$ is the valency of $\Gamma$ and $n = |V\Gamma|$.

Strongly regular graphs can also be treated as extremal graphs and have been studied extensively.
Examples

5-cycle is SRG(5, 2, 0, 1),
the Petersen graph is SRG(10, 3, 0, 1).

What are the trivial examples?

\[ K_n, \quad m \cdot K_n, \]

The **Cocktail Party graph** \( C(n) \), i.e., the graph
on \( 2n \) vertices of degree \( 2n - 2 \), is also strongly regular.
Lemma. A strongly regular graph $\Gamma$ is disconnected iff $\mu = 0$.

If $\mu = 0$, then each component of $\Gamma$ is isomorphic to $K_{k+1}$ and we have $\lambda = k - 1$.

Corollary. A complete multipartite graph is strongly regular iff its complement is a union of complete graphs of equal size.

Homework: Determine all SRG with $\mu = k$. 
Counting the edges between the neighbours and non-neighbours of a vertex in a connected strongly regular graph we obtain:

$$\mu(n - 1 - k) = k(k - \lambda - 1),$$

i.e.,

$$n = 1 + k + \frac{k(k - \lambda - 1)}{\mu}.$$  

**Lemma.** The complement of $\text{SRG}(n, k, \lambda, \mu)$ is again strongly regular graph:

$\text{SRG}(\overline{n}, \overline{k}, \overline{\lambda}, \overline{\mu}) = (n, n-k-1, n-2k+\mu-2, n-2k+\lambda).$
Let $J$ be the all-one matrix of dim. $(n \times n)$. A graph $\Gamma$ on $n$ vertices is strongly regular if and only if its adjacency matrix $A$ satisfies
\[ A^2 = kI + \lambda A + \mu(J - I - A), \]
for some integers $k$, $\lambda$ and $\mu$.

Therefore, the valency $k$ is an eigenvalue with multiplicity 1 and the nontrivial eigenvalues, denoted by $\sigma$ and $\tau$, are the roots of
\[ x^2 - (\lambda - \mu)x + (\mu - k) = 0, \]
and hence $\lambda - \mu = \sigma + \tau$, $\mu - k = \sigma \tau$. 
Theorem. A connected regular graph with precisely three eigenvalues is strongly regular.

Proof. Consider the following matrix polynomial:

\[ M := \frac{(A - \sigma)(A - \tau)}{(k - \sigma)(k - \tau)} \]

If \( A = A(\Gamma) \), where \( \Gamma \) is a connected \( k \)-regular graph with eigenvalues \( k, \sigma \) and \( \tau \), then all the eigenvalues of \( M \) are 0 or 1. But all the eigenvectors corresponding to \( \sigma \) and \( \tau \) lie in \( \text{Ker}(A) \), so rank\( M = 1 \) and \( M \mathbf{j} = \mathbf{j} \), hence \( M = \frac{1}{n} \mathbf{J} \). and \( A^2 \in \text{span}\{\mathbf{I}, \mathbf{J}, A\} \). \qed
For a connected graph, i.e., $\mu \neq 0$, we have

$$n = \frac{(k-\sigma)(k-\tau)}{k + \sigma \tau}, \quad \lambda = k + \sigma + \tau + \sigma \tau, \quad \mu = k + \sigma \tau$$

and the multiplicities of $\sigma$ and $\tau$ are

$$m_\sigma = \frac{(n-1)\tau + k}{\tau - \sigma} = \frac{(\tau + 1)k(k - \tau)}{\mu(\tau - \sigma)}$$

and $m_\tau = n - 1 - m_\sigma$. 
Multiplicities

Solve the system:

\[ 1 + m_\sigma + m_\tau = n \]
\[ 1 \cdot k + m_\sigma \cdot \sigma + m_\tau \cdot \tau = 0. \]

to obtain

\[ m_\sigma \text{ and } m_\tau = \frac{1}{2} \left( n - 1 \pm \frac{(n - 1)(\mu - \lambda) - 2k}{\sqrt{(\mu - \lambda)^2 + 4(k - \mu)}} \right). \]