Hadamard matrices

Let A be $n \times n$ matrix with $|a_{ij}| \leq 1$.

How large can $\det A$ be?

Since each column of A is a vector of length at most $\sqrt{n}$, we have

$$\det A \leq n^{n/2}.$$

Can equality hold? In this case all entries must be ± 1 and any two distinct columns must me orthogonal.

$(n \times n)$-dim. matrix H with elements ± 1, for which

$$HH^T = nI_n$$

holds is called a **Hadamard matrix** of order n.

Such a matrix exists only if $n = 1$, $n = 2$ or $4 \mid n$.

A famous **Hadamard matrix conjecture** (1893): a Hadamard matrix of order $4s$ exists $\forall s \in \mathbb{N}$.

In 2004 Iranian mathematicians H. Kharaghani and B. Tayfeh-Rezaie constructed a Hadamard matrix of order 428. The smallest open case is now 668.

$$n = 2: \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad n = 4: \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$n = 8: \quad \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & - & - & - & + & - & + \\ + & + & + & - & - & - & + & - \\ + & - & + & + & - & - & - & + \\ + & + & - & + & + & - & - & - \\ + & - & + & - & + & + & - & - \\ + & - & - & + & - & + & + & - \\ + & - & - & - & + & - & + & + \end{pmatrix}$$

Hadamard matrix of order $4s$ is equivalent to 2-$(4s-1, 2s-1, s-1)$ design.