Graphs

A **graph** Γ is a pair $(V\Gamma, E\Gamma)$, where $V\Gamma$ is a finite set of **vertices** and $E\Gamma$ is a set of unordered pairs xy of vertices called **edges** (no loops or multiple edges).

Let $V\Gamma = \{1, \ldots, n\}$. Then a $(n \times n)$-dim. matrix A is the **adjacency matrix** of Γ, when

$$A_{i,j} = \begin{cases} 1, & \text{if } \{i,j\} \in E, \\ 0, & \text{otherwise} \end{cases}$$

Lemma. $(A^h)_{ij} = \#$ *walks from i to j of length h.*

Eigenvalues

The number $\theta \in \mathbb{R}$ is an **eigenvalue** of Γ, when for a vector $x \in \mathbb{R}^n \backslash \{0\}$ we have

$$Ax = \theta x, \quad \text{i.e.,} \quad (Ax)_i = \sum_{\{j,i\} \in E} x_j = \theta x_i.$$

- There are cospectral graphs
- A triangle inequality implies that the maximum degree of a graph Γ, denoted by $\Delta(\Gamma)$, is greater or equal to $|\theta|$, i.e.,

$$\Delta(\Gamma) \geq |\theta|.$$

A graph with **precisely one** eigenvalue is a graph with one vertex, i.e., a graph with diameter **0**.

A graph with **two** eigenvalues is the complete graph K_n, $n \geq 2$, i.e., the graph with diameter **1**.

Theorem. *A connected graph of diameter d has at least $d + 1$ distinct eigenvalues.*

Review of basic matrix theory

Lemma. *Let A be a real symetric matrix. Then*

- *its eigenvalues are real numbers, and*
- *the eigenvectors corresponding to distinct eigenvalues, then they are orthogonal.*
- *If U is an A-invariant subspace of $\mathbb{R}^n$, then $U^\perp$ is also A-invariant.*
- *$\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of A.*
- *There are matrices L and D, such that*

$$L^T L = LL^T = I \quad \text{and} \quad LAL^T = D,$$

where D is a diagonal matrix of eigenvalues of A.

Lemma. *The eigenvalues of a disconnected graph are just the eigenvalues of its components and their multiplicities are sums of the corresponding multiplicities in each component.*

Regularity

A graph is **regular**, if each vertex has the same number of neighbours.

Set $\boldsymbol{j}$ to the be all-one vector in $\mathbb{R}^n$.

Lemma. *A graph is regular iff* $\boldsymbol{j}$ *is its eigenvector.*

Lemma. *If* Γ *is a regular graph of valency* k*, then the multiplicity of* k *is equal to the number of connected components of* Γ*,*
and the multiplicity of $-k$ *is equal to the number of bipartite components of* Γ*.*

Lemma. *Let Γ be a k-regular graph on n vertices with eigenvalues $k, \theta_2, \ldots, \theta_n$. Then Γ and $\overline{\Gamma}$ have the same eigenvectors, and the eigenvalues of $\overline{\Gamma}$ are $n-k-1$, $-1-\theta_2, \ldots, -1-\theta_n$.*

Calculate the eigenvalues of many simple graphs:

- $m * K_n$ and their complements,
- circulant graphs
- C_n,
- $K_n \times K_n$,
- Hamming graphs,...

Line graphs and their eigenvalues

Let $\boldsymbol{\phi(\Gamma, x)}$ be the **characteristics polynomial** of a graph Γ.

Lemma. *Let B be the incidence matrix of the graph Γ, L its line graph and $\Delta(\Gamma)$ the diagonal matrix of valencies. Then*

$$B^T B = 2I + A(L) \quad \text{and} \quad BB^T = \Delta(\Gamma) + A(\Gamma).$$

Furthermore, if Γ is k-regular, then

$$\phi(L, x) = (x+2)^{e-n} \phi(\Gamma, x-k+2).$$

Semidefinitness

A real symmetric matrix A is **positive semidefinite** if

$$u^T Au \geq 0 \qquad \text{for all vectors } u.$$

It is **positive definite** if it is positive semidefinite and

$$u^T Au = 0 \iff u = 0.$$

Characterizations.

- *A positive semidefinite matrix is positive definite iff invertible*
- *A matrix is positive semidefinite matrix iff all its eigenvalues are nonnegative.*
- *If $A = B^T B$ for some matrix, then A is positive semidefinite.*

The **Gram matrix** of vectors $u_1, \dots, u_n \in \mathbb{R}^m$ is $n \times n$ matrix G s.t. $G_{ij} = u_i^t u_j$.

Note that $B^T B$ is the Gram matrix of the columns of B, and that any Gram matrix is positive semidefinite. The converse is also true.

Corollary. *The least eigenvalue of a line graph is at least -2. If Δ is an induced subgraph of Γ, then*

$$\theta_{\min}(\Gamma) \le \theta_{\min}(\Delta) \le \theta_{\max}(\Delta) \le \theta_{\max}(\Gamma).$$

Peron-Frobenious Theorem. *Suppose* A *is a real nonnegative* $n \times n$ *matrix, whose underlying directed graph* X *is strongly connected. Then*

(a) $\rho(A)$ *is a simple eigenvalue of* A*. If* x *an eigenvector for* ρ*, then no entries of* x *are zero, and all have the same sign.*

(b) *Suppose* A_1 *is a real nonnegative* $n \times n$ *matrix such that* $A - A_1$ *is nonnegative. Then* $\rho(A_1) \leq \rho(A)$*, with equality iff* $A_1 = A$*.*

(c) *If* θ *is an eigenvalue of* A *and* $|\theta| = \rho(A)$*, then* $\theta/\rho(A)$ *is an* m*th root of unity and* $e^{2\pi i r/m}\rho(A)$ *is an eigenvalue of* A *for all* r*. Further, all cycles in* X *have length divisible by* m*.*

Theorem [Haemers]. *Let A be a complete hermitian $n \times n$ matrix, partitioned into m^2 block matrices, such that all diagonal matrices are square. Let B be the $m \times m$ matrix, whose i, j-th entry equals the average row sum of the i, j-th block matrix of A for $i, j = 1, \dots, m$. Then the eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_m$ of A and B resp. satisfy*

$$\boldsymbol{\alpha_i \geq \beta_i \geq \alpha_{i+n-m}, \quad for \;\; i = 1, \dots, m.}$$

Moreover, if for some $k \in N_0$, $k \leq m$, $\alpha_i = \beta_i$ for $i = 1, \dots, k$ and $\beta_i = \alpha_{i+n-m}$ for $i = k+1, \dots, m$, then all the block matrices of A have constant row and column sums.