Algebraic Combinatorics, 2007

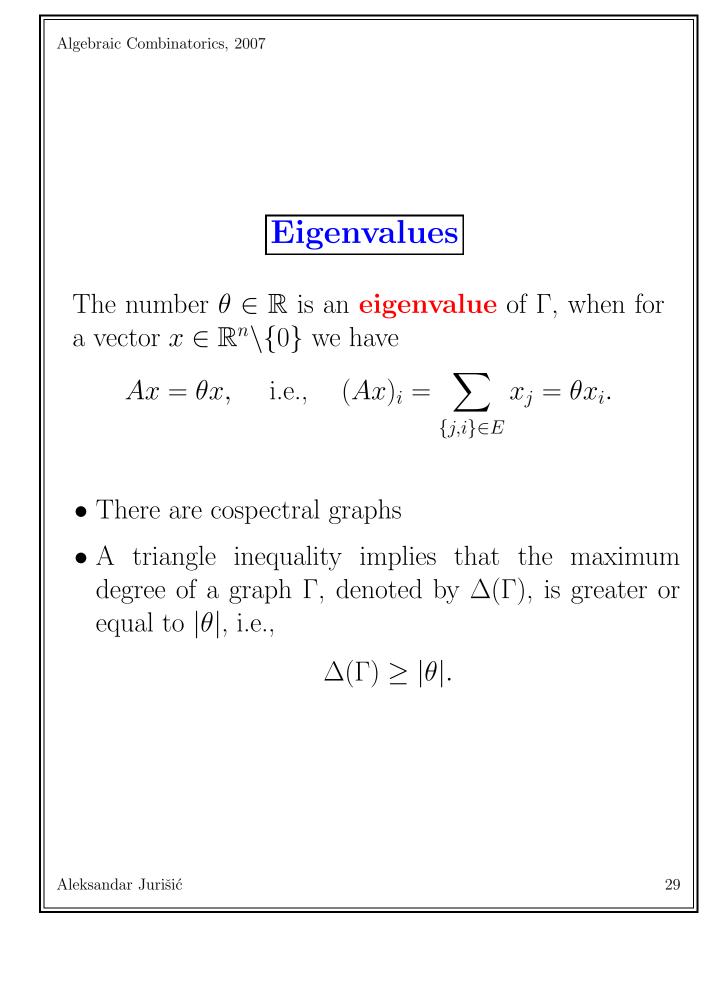
Graphs

A graph Γ is a pair $(V\Gamma, E\Gamma)$, where $V\Gamma$ is a finite set of **vertices** and $E\Gamma$ is a set of unordered pairs xyof vertices called **edges** (no loops or multiple edges).

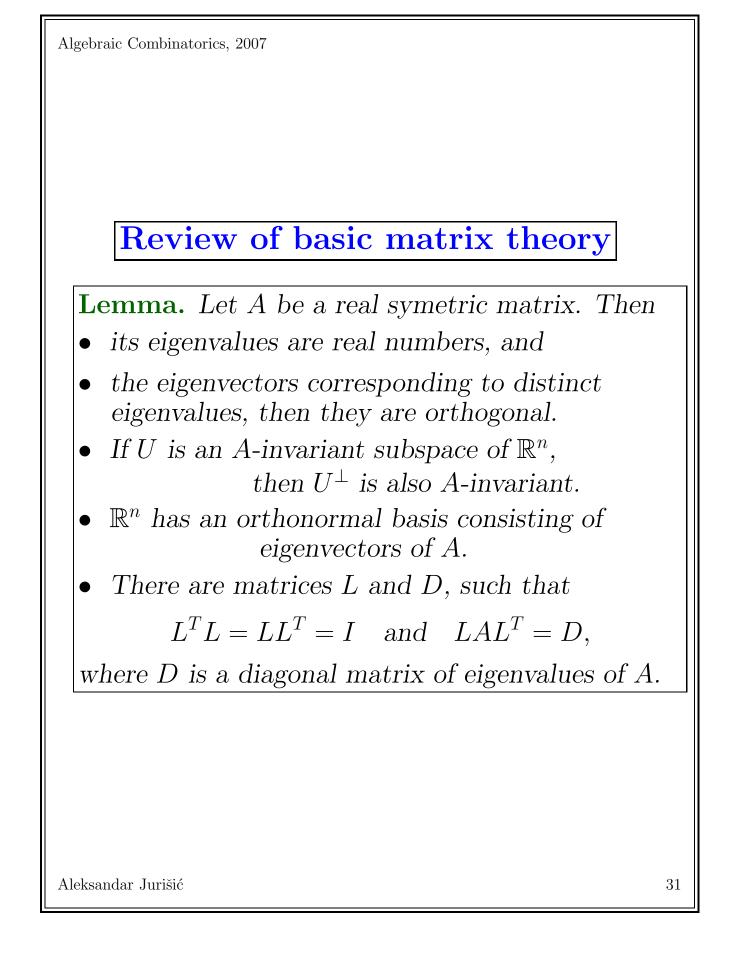
Let $V\Gamma = \{1, \ldots, n\}$. Then a $(n \times n)$ -dim. matrix A is the **adjacency matrix** of Γ , when

 $A_{i,j} = \begin{cases} 1, \text{ if } \{i,j\} \in E, \\ 0, \text{ otherwise} \end{cases}$

Lemma. $(A^h)_{ij} = \#$ walks from *i* to *j* of length *h*.



Algebraic Combinatorics, 2007
A graph with precisely one eigenvalue is a graph with one vertex, i.e., a graph with diameter 0 .
A graph with two eigenvalues is the complete graph $K_n, n \ge 2$, i.e., the graph with diameter 1 .
Theorem. A connected graph of diameter d has at least $d + 1$ distinct eigenvalues.



Algebraic Combinatorics, 2007	
Lemma. The eigenvalues of a disconnect	ted graph
are just the eigenvalues of its component	
their multiplicities are sums of the corres	
multiplicities in each component.	sponding
Aleksandar Jurišić	32

Algebraic Combinatorics, 2007

Regularity

A graph is **regular**, if each vertex has the same number of neighbours.

Set \mathbf{j} to the be all-one vector in \mathbb{R}^n .

Lemma. A graph is regular iff j is its eigenvector.

Lemma. If Γ is a regular graph of valency k, then the multiplicity of k is equal to the number of connected components of Γ , and the multiplicity of -k is equal to the number of bipartite components of Γ .

Lemma. Let Γ be a k-regular graph on n vertices with eigenvalues $k, \theta_2, \ldots, \theta_n$. Then Γ and $\overline{\Gamma}$ have the same eigenvectors, and the eigenvalues of $\overline{\Gamma}$ are $n - k - 1, -1 - \theta_2, \ldots, -1 - \theta_n$.

Calculate the eigenvalues of many simple graphs:

- $m * K_n$ and their complements,
- circulant graphs
- C_n ,
- $K_n \times K_n$,
- Hamming graphs,...

Line graphs and their eigenvalues

Let $\phi(\Gamma, x)$ be the characteristics polynomial of a graph Γ .

Lemma. Let B be the incidence matrix of the graph Γ , L its line graph and $\Delta(\Gamma)$ the diagonal matrix of valencies. Then

 $B^T B = 2I + A(L)$ and $BB^T = \Delta(\Gamma) + A(\Gamma)$. Furthermore, if Γ is k-regular, then $\phi(L, x) = (x+2)^{e-n}\phi(\Gamma, x-k+2).$

Semidefinitness

A real symmetric matrix A is **positive semidefinite** if

 $u^T A u \ge 0$ for all vectors u.

It is **positive definite** if it is positive semidefinite and

$$u^T A u = 0 \iff u = 0.$$

Characterizations.

- A positive semidefinite matrix is positive definite iff invertible
- A matrix is positive semidefinite matrix iff all its eigenvalues are nonnegative.
- If $A = B^T B$ for some matrix, then A is positive semidefinite.

The **Gram matrix** of vectors $u_1, \ldots, u_n \in \mathbb{R}^m$ is $n \times n$ matrix G s.t. $G_{ij} = u_i^t u_j$.

Note that $B^T B$ is the Gram matrix of the columns of B, and that any Gram matrix is positive semidefinite. The converse is also true.

Corollary. The least eigenvalue of a line graph is at least -2. If Δ is an induced subgraph of Γ , then $\theta_{\min}(\Gamma) \leq \theta_{\min}(\Delta) \leq \theta_{\max}(\Delta) \leq \theta_{\max}(\Gamma).$

Peron-Frobenious Theorem. Suppose A is a real nonnegative $n \times n$ matrix, whose underlying directed graph X is strongly connected. Then

(a) $\rho(A)$ is a simple eigenvalue of A. If x an eigenvector for ρ , then no entries of x are zero, and all have the same sign.

(b) Suppose A_1 is a real nonnegative $n \times n$ matrix such that $A - A_1$ is nonnegative. Then $\rho(A_1) \leq \rho(A)$, with equality iff $A_1 = A$.

(c) If θ is an eigenvalue of A and $|\theta| = \rho(A)$, then $\theta/\rho(A)$ is an *m*th root of unity and $e^{2\pi i r/m}\rho(A)$ is an eigenvalue of A for all r. Further, all cycles in X have length divisible by m.

Theorem [Haemers]. Let A be a complete hermitian $n \times n$ matrix, partitioned into m^2 block matrices, such that all diagonal matrices are square. Let B be the $m \times m$ matrix, whose i, j-th entry equals the average row sum of the i, j-th block matrix of A for i, j = 1, ..., m. Then the eigenvalues $\alpha_1 \ge \cdots \ge \alpha_n$ and $\beta_1 \ge \cdots \ge \beta_m$ of A and B resp. satisfy

 $\alpha_i \geq \beta_i \geq \alpha_{i+n-m}, \quad for \quad i = 1, \dots, m.$ Moreover, if for some $k \in N_0, k \leq m, \alpha_i = \beta_i$ for $i = 1, \dots, k$ and $\beta_i = \alpha_{i+n-m}$ for $i = k+1, \dots, m,$ then all the block matrices of A have constant row and column sums.