Let $\mathcal{O}$ be a subset of points of $\mathrm{PG}(2, n)$ such that no three are on the same line.

Then $|\mathcal{O}| \leq n + 1$ if n is odd
and $|\mathcal{O}| \leq n + 2$ if n is even.

If equality is attained then $\mathcal{O}$ is called
oval for n even, and **hyperoval** for n odd

Examples:

- the vertices of a triangle and the center of the circle in Fano plane,
- the vertices of a square in $PG(2,3)$ form oval,
- the set of vertices $\{0,1,2,3,5,14\}$ in the above $PG(2,4)$ is a hyperoval.

The **general linear group** $\mathrm{GL}_n(q)$ consists of all invertible $n \times n$ matrices with entries in $\mathrm{GF}(q)$.

The **special linear group** $\mathrm{SL}_n(q)$ is the subgroup of all matrices with determinant 1.

The **projective general linear group** $\mathrm{PGL}_n(q)$ and the **projective special linear group** $\mathrm{PSL}_n(q)$ are the groups obtained from $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$ by taking the quotient over scalar matrices (i.e., scalar multiple of the identity matrix).

For $n \geq 2$ the group $\mathrm{PSL}_n(q)$ is simple
(except for $\mathrm{PSL}_2(2) = S_3$ and $\mathrm{PSL}_2(3) = A_4$)
and is by Artin's convention denoted by $L_n(q)$.

Orthogonal Arrays

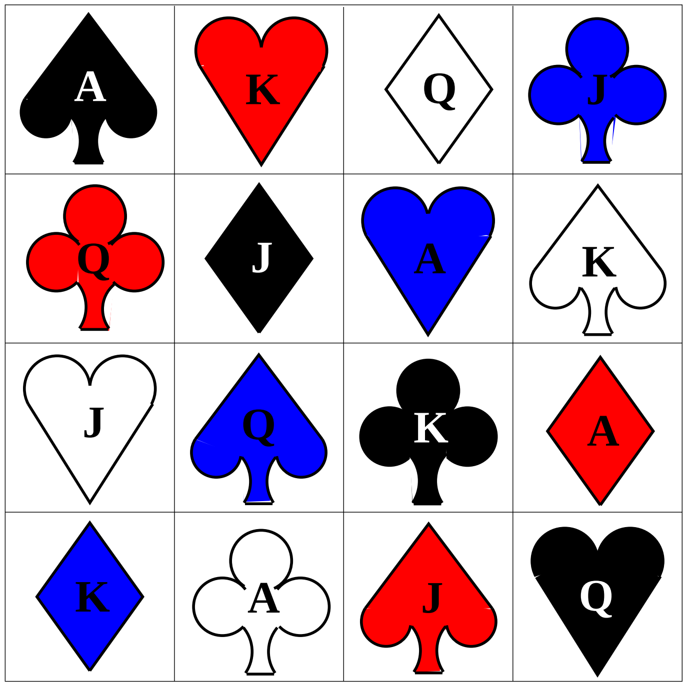
An **orthogonal array**, OA(v, s, λ), is such $(\lambda v^2 \times s)$-dimensional matrix with v symbols, that each two columns each of v^2 possible pairs of symbols appears in exactly λ rows.

This and to them equivalent structures (e.g. transversal designs, pairwise orthogonal Latin squares, nets,...) are part of design theory.

If we use the first two columns of $\mathrm{OA}(v, s, 1)$ for coordinates, the third column gives us a **Latin square**, i.e., $(v \times v)$-dim. matrix in which all symbols $\{1, \dots, v\}$ appear in each row and each column.

Example : $\mathrm{OA}(3,3,1)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$



Three pairwise orthogonal Latin squares of order 4, i.e., each pair symbol-letter or letter-color or color-symbol appears exactly once.

Theorem. *If* $\mathrm{OA}(v, s, \lambda)$ *exists, then we have in the case* $\lambda = 1$

$$s \leq v + 1,$$

and in general

$$\lambda \geq \frac{s(v-1)+1}{v^2}.$$

Transversal design $\mathrm{TD}_\lambda(s, v)$ is an incidence structure of blocks of size s, in which points are partitioned into s groups of size v so that an arbitrary points lie in λ blocks when they belong to distinct groups and there is no block containing them otherwise.

Proof: The number of all lines that intersect a chosen line of $\mathrm{TD}_1(s,v)$ is equal to $(v-1)s$ and is less or equal to the number of all lines without the chosen line, that is v^2-1.

In transversal design $\mathrm{TD}_\lambda(s,v)$, $\lambda \neq 1$ we count in a similar way and then use the inequality between arithmetic and quadratic mean (that can be derived from Jensen inequality). ■

Theorem. *For a prime p there exists* $\mathrm{OA}(p,p,1)$, *and there also exists* $\mathrm{OA}(p,(p^d-1)/(p-1),p^{d-2})$ *for* $d \in \mathbb{N}\backslash\{1\}$

Proof: Set $\lambda = 1$. For $i, j, s \in \mathbb{Z}_p$ we define

$$e_{ij}(s) = is + j \bmod p.$$

For $\lambda \neq 1$ we can derive the existence from the construction of projective geometry $\mathrm{PG}(n,d)$. ■

For homework convince yourself that each $\mathrm{OA}(n,n,1)$, $n \in \mathbb{N}$, can be extended for one more column, i.e., to $\mathrm{OA}(n,n+1,1)$.