Extremal 1-codes in distance-regular graphs of diameter 3

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Abstract

We study 1-codes in distance-regular graphs of diameter 3 that achieve three different bounds. We show that the intersection array of a distance-regular graph containing such a code has the form

$$\{a(p+1), cp, a+1; 1, c, ap\} \quad \text{or} \quad \{a(p+1), (a+1)p, c; 1, c, ap\}$$

for $c = c_2$, $a = a_3$ and $p = p^3_{33}$. These two families contain $10+15$ known feasible intersection arrays out of which four are uniquely realized by the Sylvester graph, the Hamming graph $H(3,3)$, the Doro graph and the Johnson graph $J(9,3)$, but not all members of these two families are feasible. We construct four new one-parameter infinite subfamilies of feasible intersection arrays, two of which have a nontrivial vanishing Krein parameter:

$$\{(2r^2-1)(2r+1), 4r(r^2-1), 2r^2; 1, 2(r^2-1), r(4r^2-2)\}$$

and

$$\{2r^2(2r+1), (2r-1)(2r^2+r+1), 2r^2; 1, 2r^2, r(4r^2-1)\}$$

for $r > 1$ (the second family actually generalizes to a two-parameter family with the same property). Using this information we calculate some triple intersection numbers for these two families to show that they must contain the desired code. Finally, we use some additional combinatorial arguments to prove nonexistence of distance-regular graphs with such intersection arrays.

Keywords: distance-regular graphs, 1-codes, Krein condition, triple intersection numbers, nonexistence, algebraic combinatorics

1 Introduction

Using the distance function, we study codes in distance-regular graphs. A number of bounds for codes can be obtained. The best known one is the sphere packing bound. In this paper we will be looking at codes in distance-regular graphs of diameter 3 that achieve three different bounds. We show that the intersection array of a distance-regular graph containing such a code has the form

$$\{a(p+1), cp, a+1; 1, c, ap\} \qquad \text{and} \qquad \{a(p+1), (a+1)p, c; 1, c, ap\} \qquad \text{for } c = c_2,\ a = a_3 \text{ and } p = p^3_{33}.$$

They are satisfied by 10 and 15 feasible intersection array from the table of feasible intersection arrays for primitive graphs with diameter 3 from Brouwer, Cohen and Neumaier [1, pp. 425–431], respectively. However, not all members of these two families are feasible. We then construct four new infinite subfamilies of feasible intersection arrays:

$$\{(2r^2-1)(2r+1), 4r(r^2-1), 2r^2; 1, 2(r^2-1), r(4r^2-2)\}, \qquad r > 1; \tag{1}$$

$$\{c^2-1, c(c-2), c+2; 1, c, (c+1)(c-2)\}, \qquad c \geq 6; \tag{2}$$

$$\{2r^2(2r+1), (2r-1)(2r^2+r+1), 2r^2; 1, 2r^2, r(4r^2-1)\}, \qquad r \geq 1; \tag{3}$$

$$\{2r(r+1)(2r+1), 2r(2r^2+2r+1), r(2r+1); 1, r(2r+1), 4r^2(r+1)\}, \quad r \geq 1. \tag{4}$$

The first cases of (3) and (4) ($r = 1$) have intersection arrays $\{6, 4, 2; 1, 2, 3\}$ and $\{12, 10, 3; 1, 3, 8\}$ and are known to uniquely determine the Hamming graph $H(3, 3)$ [1, p. 262] and the Doro graph [1, Sec. 12.1], respectively. For (1) and (3), the graphs with such intersection arrays have a nontrivial vanishing Krein parameter. Therefore, we can use the method of Coolsaet and Jurišić [2] to calculate some triple intersection numbers. This way we prove that such graphs indeed contain the desired codes (the family (3) actually generalizes to a two-parameter family with the same property). We then use some additional combinatorial arguments to prove nonexistence of distance-regular graphs with such intersection arrays (except for $r = 1$ in the case of (3)). The first two cases of (1) ($r = 2, 3$) are $\{35, 24, 8; 1, 6, 28\}$ and $\{119, 96, 18; 1, 16, 102\}$ and the second case of (3) ($r = 2$) is $\{40, 33, 8; 1, 8, 30\}$ and they all appear in the table from Brouwer et al. [1, pp. 425–431]. Additional information about the graphs is shown in Table 1.

In Section 3 we first give some definitions and results on codes in distance-regular graphs. For a distance-regular graph Γ with intersection array (1) or (3), Section 4 lists certain parameters that will be used in the following sections. In Section 5 we then prove that for any vertices u and v of Γ at distance 3, the set that consists of u, v and the vertices of Γ that are at distance 3 from both of them is a maximal 1-code. We show in Section 6 that Γ does not exist. We conclude the paper with Section 7 that presents some open questions on the subject of codes in distance-regular graphs.

graph	n	θ_1	θ_2	θ_3	m_1	m_2	m_3
(1)	$4r(2r-1)(r+1)^2$	$2r^2+2r-1$	-1	$-2r^2+1$	$(2r^2-1)(2r+1)$	$2r(2r^2-1)(2r+1)$	$2r^2(2r+1)$
(2)	c^3	$2c-1$	-1	$-c-1$	$c(c^2-1)/6$	$(c+1)(c^2+c-2)/2$	$c(c-1)(c-2)/3$
(3)	$r(2r+1)^3$	$r(2r+1)$	0	$-r(2r+1)$	$2r^2(2r+1)$	$(4r^2-1)(2r^2+r+1)$	$2r(2r^2+r+1)$
(4)	$2(r+1)(4r(r+1)^2+1)$	$2r(r+1)$	0	$-2r^2-2r-1$	$4r(r+1)^2+1$	$2r(4r(r+1)^2+1)$	$4(r+1)^2$

Table 1: The number of vertices n and spectrum $\{\theta_0^1, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3}\}$ of the distance-regular graphs with intersection arrays (1–4), where $k = \theta_0 > \theta_1 > \theta_2 > \theta_3$.

2 Preliminaries

In this section we review some basic definitions and concepts. See Brouwer, Cohen and Neumaier [1] and Godsil [3] for further details.

Let Γ be a finite, undirected, connected graph, without loops or multiple edges, with vertex set $V\Gamma$, edge set $E\Gamma$, shortest path-length distance function ∂, and diameter $d := \max\{\partial(x, y) \mid x, y \in V\Gamma\}$. For vertices $u_1, \dots, u_t \in V\Gamma$ and integers $r_1, \dots, r_t \in \{0, 1, \dots, d\}$ define

$$\left\{ \begin{smallmatrix} u_1 & \dots & u_t \\ r_1 & \dots & r_t \end{smallmatrix} \right\} := \{v \in V\Gamma \mid \partial(v, u_i) = r_i, i = 1, \dots, t\} \quad \text{and} \quad \left[\begin{smallmatrix} u_1 & \dots & u_t \\ r_1 & \dots & r_t \end{smallmatrix} \right] := \left| \left\{ \begin{smallmatrix} u_1 & \dots & u_t \\ r_1 & \dots & r_t \end{smallmatrix} \right\} \right|, \tag{5}$$

where $|S|$ denotes the cardinality of a set S. Note that the symbols defined in (5) are invariant under permutations of their columns. We will abbreviate the latter as $[r_1 \ \dots \ r_t]$ whenever no confusion about the t-tuple $(u_1, \dots, u_t)$ may arise. When $t = 3$, we call the resulting cardinalities **triple intersection numbers**.

Let $\Gamma_i(u)$ for some vertex $u \in V\Gamma$ and an integer i be the subgraph of Γ that is induced by the vertices at distance i from u. The **local graph** $\Delta(u)$ for some vertex $u \in V\Gamma$ is the graph induced by the neighbours of the vertex u in the graph Γ, i.e. $\Delta(u) = \Gamma_1(u)$. We also define the **distance i graph** Γ_i as the graph with $V\Gamma_i = V\Gamma$ and $u \sim v$ in Γ_i whenever $\partial(u, v) = i$ in Γ.

For an integer $k \geq 0$, the graph Γ is said to be **regular with valency k** whenever $\left[\begin{smallmatrix} u \\ 1 \end{smallmatrix} \right] = k$ for all $u \in V\Gamma$. The graph Γ is said to be **distance-regular** whenever for $0 \leq i, j, h \leq d$ the number $p^h_{ij} = \left[\begin{smallmatrix} u & v \\ i & j \end{smallmatrix} \right]$ is independent of u, v for all vertices $u, v \in V\Gamma$ with $\partial(u, v) = h$, The constants p^h_{ij} are called the **intersection numbers** of Γ. Let $i \in \{0, 1, \dots, d\}$. We define $c_i := p^i_{1,i-1}$, $a_i := p^i_{1i}$, $b_i := p^i_{1,i+1}$, $k_i := p^0_{ii}$, where we assumed $p^0_{1,-1} = p^d_{1,d+1} = 0$. Moreover, $a_0 = c_0 = b_d = 0$, $c_1 = 1$ and $a_i + b_i + c_i = k$, where $k := k_1$. Note that the distance-regularity immediately implies $\left[\begin{smallmatrix} u \\ i \end{smallmatrix} \right] = k_i$ for any $u \in V\Gamma$.

Let Γ be a graph of diameter d. Let A_i $(0 \leq i \leq d)$ denote an adjacency matrix of Γ_i, i.e., the binary matrix indexed by the vertices of the graph Γ, where $A_i(u, v) = 1$ iff $\partial(u, v) = i$ for any $u, v \in V\Gamma$. Set $A := A_1$. Let us assume additionally that Γ is distance-regular. Then A has precisely $d + 1$ distinct eigenvalues $k = \theta_0, \theta_1, \dots, \theta_d$ and $\theta_i < \theta_0$ $(1 \leq i \leq d)$ [1, p. 128]. Let m_i $(0 \leq i \leq d)$ denote the multiplicity of the eigenvalue θ_i. Let $\mathcal{M}$ be the Bose-Mesner algebra, that is the algebra generated by A. Then the matrices $\{A_i\}_{i=0}^d$ form a base of $\mathcal{M}$. There is also a base of minimal idempotents $\{E_i\}_{i=0}^d$ [1, p. 44] with the property $\sum_{i=0}^d E_i = I$, $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq d)$ and $|V\Gamma| E_0 = J$, where J is the all-one matrix. We define matrices P and Q such that $A_j = \sum_{i=0}^d P_{ij} E_i$ and $E_j = |V\Gamma|^{-1} \sum_{i=0}^d Q_{ij} A_i$ $(0 \leq j \leq d)$. Note that there is a correspondence between these eigenvalues and minimal idempotents and that $\{P_{ij}\}_{i=0}^d$ are the eigenvalues of A_j. The matrices P and Q are called the **eigenmatrix** and the **dual eigenmatrix** of Γ. The ordering of minimal idempotents corresponding to the descending order of eigenvalues is known as the **natural ordering**. A graph Γ is called **formally self-dual** [1, p. 49] when $P = Q$ for some ordering of the minimal idempotents. A graph Γ is called **Q-polynomial** with respect to the minimal idempotent E [1, p. 58] for some ordering of the minimal idempotents such that $E = E_1$, if there exist real numbers $z_0, \dots, z_d$ and polynomials q_j of degree j such that $Q_{ij} = q_j(z_i)$ $(0 \leq i, j \leq d)$. A distance-regular graph that is formally self-dual is also Q-polynomial with respect to the minimal idempotent E_1 for an ordering that achieves $P = Q$. Furthermore, we can define the **Krein parameters** q^h_{ij} [1, p. 48] as such numbers that $E_i \circ E_j = |V\Gamma|^{-1} \sum_{h=0}^d q^h_{ij} E_h$ $(0 \leq i, j \leq d)$, where $\circ$ represents entrywise multiplication of matrices. If Γ is formally self-dual, then $q^h_{ij} = p^h_{ij}$ $(0 \leq h, i, j \leq d)$.

Unlike for the cases $t = 1$ and $t = 2$, for $t \geq 3$ there are no formulas for $[r_1 \ \dots \ r_t]$ that are generally valid in the case of distance-regular graphs. However, for the case $t = 3$ that we are interested in, certain restrictions for their values may be found sometimes. Let $u, v, w \in V\Gamma$ be three fixed vertices in $V\Gamma$, and let U, V and W be the distances between them, i.e., $\partial(u,v) = W$, $\partial(v,w) = U$ and $\partial(w,u) = V$. Then there exists precisely one vertex $x = u$ such that $\partial(x,u) = 0$, so $[0\ j\ h]$ is either 0 or 1. We can apply the same argument also for v and w. Altogether, we obtain

$$[0\ j\ h] = \delta_{jW}\delta_{hV}, \qquad [i\ 0\ h] = \delta_{iW}\delta_{hU}, \qquad [i\ j\ 0] = \delta_{iV}\delta_{jU} \qquad (0 \le i,j,h \le d). \tag{6}$$

Another set of equations can be obtained by fixing the distance from two of the vertices u, v, w and counting vertices at all distances from the third vertex:

$$\sum_{\ell=1}^{d} [\ell\ j\ h] = p^U_{jh} - [0\ j\ h], \qquad \sum_{\ell=1}^{d} [i\ \ell\ h] = p^V_{ih} - [i\ 0\ h], \qquad \sum_{\ell=1}^{d} [i\ j\ \ell] = p^W_{ij} - [i\ j\ 0]. \tag{7}$$

The system of $3d^2$ equations (7) has d^3 integral nonnegative variables $[i\ j\ h]$ $(1 \le i,j,h \le d)$. These equations are not all linearly independent. However, we can use the triangle inequality to conclude vanishing of some variables. For example, for $0 \le i,j \le d$ and $|i-j| > W$ or $i + j < W$ we have $p^W_{ij} = 0$ and so also $[i\ j\ h] = 0$ $(0 \le h \le d)$. Usually, there is no single solution, as the number of variables is generally greater than the number of linearly independent equations, yet some additional information about the structure of the distance-regular graph can sometimes be obtained. This will be useful in proving Lemmas 5.1, 6.2, 6.3 and Theorem 6.1.

If a Krein parameter q^h_{ij} is zero, we can obtain another equation for triple intersection numbers.

Theorem 2.1. ([1, Theorem 2.3.2], [2, Theorem 3]) *Consider a distance-regular graph with diameter d, dual eigenmatrix Q and Krein parameters q^h_{ij} for $i,j,h \in \{0,1,\dots,d\}$. For vertices $u,v,w \in V\Gamma$ define*

$$S_{ijh}(u,v,w) := \sum_{r,s,t=0}^{d} Q_{ri}Q_{sj}Q_{th}\begin{bmatrix} u & v & w \\ r & s & t \end{bmatrix}.$$

Then $S_{ijh}(u,v,w) = 0$ whenever $q^h_{ij} = q^j_{ih} = q^i_{jh} = 0$.

The identity $S_{ijh}(u,v,w) = 0$ has the same set of variables as the equations (7), however it turns out to be independent of them in general (when $i,j,h \ne 0$). Furthermore, there are instances when $S_{ijh}(u,v,w) = 0$, $S_{jhi}(u,v,w) = 0$ and $S_{hij}(u,v,w) = 0$ are linearly independent as we will see in Sections 5 and 6.

3 Codes

We now introduce some results and terminology on codes. Let Γ be a graph with diameter d and e a positive integer. A subset C of the vertex set of Γ is called an ***e*-code** if its minimum distance $\delta := \min\{\partial(u,v) \mid u,v \in C,\ u \ne v\}$ is at least $2e+1$, i.e., balls of radius e around the elements of C are all pairwise disjoint. For an e-code C in a distance-regular graph, its size is bounded above, since $\left(\sum_{i=0}^{e} \begin{bmatrix} u \\ i \end{bmatrix}\right)|C| \le |V\Gamma|$ for a vertex $u \in V\Gamma$. This bound is known as the sphere packing bound [3, p. 238]. An e-code is called **perfect** if equality holds in the sphere packing bound. We will now specialize on codes in distance-regular graphs of odd diameter.

Proposition 3.1. *Let Γ be a distance-regular graph of diameter $d = 2e+1$, $e \in \mathbb{N}$, and C an e-code in Γ. Then, $|C| \le p^d_{dd} + 2$.*

Proof. Let $u, v \in C$ be two distinct vertices. Then $\partial(u,v) = d$ and the set of vertices at distance d from both u and v together with these two vertices contains C. The size of this set is $p^d_{dd} + 2$ and the result follows. $\square$

If equality is achieved in the above bound, then we call C a **maximal** e-code.

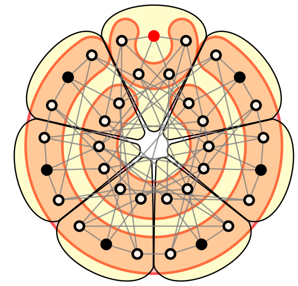


Figure 1: A perfect 1-code in the Odd graph on 7 points.

Proposition 3.2. *Let Γ be a distance-regular graph of diameter $d = 2e+1$, $e \in \mathbb{N}$, with a maximal e-code C. Then $c_d \geq a_d\, p^d_{dd}$ holds. If equality is achieved, then each vertex u that is adjacent to a vertex $v \in C$ is at distance d from precisely one vertex in C. In other words, for any $u, v, w \in V\Gamma$ with $v, w \in C$, $\partial(u, v) = 1$ and $\partial(u, w) = d - 1$, we have $\left[\begin{smallmatrix} u & v & w \\ d & d & d \end{smallmatrix}\right] = 1$.*

Proof. If $|C| = 2$, then $p^d_{dd} = 0$ and the statement is obvious. Otherwise, let $v, w, x \in C$ be three distinct vertices. As all vertices at distance d from w and x are in the code, it follows that no neighbour of v is at distance d from both w and x. So every neighbour of v is at distance d from either one or zero vertices of C. Therefore,

$$\bigcup_{w \in C \setminus \{x, v\}} \left\{\begin{matrix} v & w \\ 1 & d \end{matrix}\right\} \subseteq \left\{\begin{matrix} v & x \\ 1 & d-1 \end{matrix}\right\}$$

and since the sets on the left-hand side are disjoint, we have $a_d\, p^d_{dd} \leq c_d$. If equality holds, then there are no neighbours u of v that are at distance d from zero vertices of C and the stated properties of locally regular codes follow. $\square$

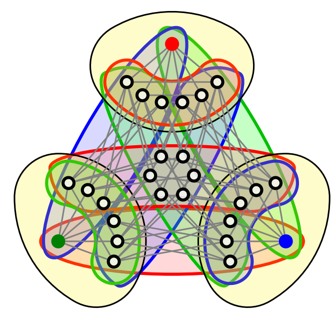


Figure 2: A locally regular and last subconstituent perfect 1-code in the Hamming graph $H(3, 3)$. This graph is also 3-homogeneous.

If equality holds in the above bound, then we call C a **locally regular** e-code.

Proposition 3.3. *Let Γ be a distance-regular graph of diameter $d = 2e+1$, $e \in \mathbb{N}$, and C an e-code in Γ. Then,*

$$|C| \leq \frac{k_d}{\sum_{i=0}^{e} p^d_{id}} + 1. \tag{8}$$

Proof. Let $u \in C$. Then, $C \setminus \{u\}$ is an e-code on $\Gamma_d(u)$, so the result follows by the sphere packing bound and the properties of perfect e-codes. $\square$

If equality is achieved in (8), then we call C a **last subconstituent perfect** e-code and each vertex v that is at distance d from a vertex $u \in C$ is at distance at most e from precisely one vertex of C.

Corollary 3.4. *Let Γ be a distance-regular graph of diameter $d = 2e + 1$, $e \in \mathbb{N}$, with a maximal e-code C. Then $k_d \geq (p^d_{dd} + 1) \sum_{i=0}^{e} p^d_{id}$ holds.*

Proof. Since C is maximal, we have $|C| = p^d_{dd} + 2$. The result then follows by Proposition 3.3. $\square$

Examples of 1-codes in distance-regular graphs of diameter 3 are shown in Figures 1 and 2. In both figures the elements of the codes are the filled vertices, with the balls of radius 1 around them outlined. For square vertices, their distance partitions are also shown.

The vertices of the Odd graph on 7 points, as shown in Figure 1, can be thought of as subsets of size 3 of a set P with 7 elements, with two vertices being adjacent whenever they are disjoint. It can easily be checked that any two vertices at distance 2 have precisely two elements in common, while any two vertices at distance 3 have precisely one common element. If we think of vertices as lines, then a perfect 1-code is achieved by choosing the lines of a

Fano plane with point set P. Note that this code is not maximal, as $p^3_{33} = 9$ and equality is not achieved in the bound from Proposition 3.1.

The vertices of the Hamming graph $H(3, 3)$, as shown in Figure 2, can be thought of vectors of length 3 over an alphabet of size 3, with two vertices being adjacent when they match in exactly two positions of the vector. A maximal 1-code can be obtained by choosing three vertices such that any two differ in all positions. This code is also locally regular and last subconstituent perfect, but it is not perfect as there are 6 vertices in the graph that are at distance 2 from all three vertices in the code (i.e., the ones that match with each code vertex in exactly one position).

Proposition 3.5. *Let Γ be a distance-regular graph of diameter $d = 2e + 1$, $e \in \mathbb{N}$, and C a perfect e-code in Γ. Then C is also a last subconstituent perfect e-code.*

Proof. Let $u \in C$. For every $x \in \Gamma_d(u)$ there is precisely one vertex $v \in C \setminus \{u\}$ such that $\partial(x, v) \leq e$. Therefore, $k_d = (|C| - 1) \sum_{i=0}^{e} p^d_{id}$ and the code C is last subconstituent perfect. □

Proposition 3.6. *Let Γ be a distance-regular graph of diameter 3 with a maximal 1-code C that is both locally regular and last subconstituent perfect. Set $a = a_3$, $p = p^3_{33}$ and $c = c_2$. Then Γ is primitive and its intersection array is either*

$$\{a(p + 1), cp, a + 1; 1, c, ap\} \qquad \text{or} \tag{9}$$

$$\{a(p + 1), (a + 1)p, c; 1, c, ap\}. \tag{10}$$

Furthermore, if Γ has intersection array (10), then it is a Shilla graph.

Proof. Since the code C is locally regular, we have $c_3 = ap$ and $k = a + c_3 = a(p + 1)$ by Proposition 3.2. Since $a = a_3 \neq 0$, the graph Γ is primitive. By [1, Lem. 4.1.7], $p^3_{23} = a(b_1 + p(b_2 - a - 1))/c$. Then $p = p^3_{33} = k - 1 - a - p^3_{23}$ gives us $b_1 = p(1 + a + c - b_2)$. Taking into account that C is also last subconstituent perfect, we obtain $k_3 = (p + 1)(a + 1)$ and therefore $b_1b_2 = cp(a + 1)$, i.e., $b_2(a + c + 1 - b_2) = c(a + 1)$. By solving the obtained quadratic equation for b_2 we derive the desired intersection arrays. Finally, we assume Γ has the second intersection array. It can easily be verified that $a = a_3$ is an eigenvalue of Γ. By [4, Thm. 7] we then have $\theta_1 = a_3$, so Γ is a Shilla graph. □

Remark 3.7. *The intersection arrays (9) and (10) are often not feasible. Table 2 lists intersection arrays from the two families that can be found in Brouwer et al. [1, pp. 425–431].*

Let Γ be a k-regular graph. If there are integers λ and μ such that the number of common neighbours of any two distinct vertices is λ when they are adjacent and μ otherwise, then Γ is said to be **strongly regular** with parameters (k, λ, μ).

Proposition 3.8. *Let Γ be a distance-regular graph with intersection array (9), eigenvalues $\theta_0 > \theta_1 > \theta_2 > \theta_3$ and a maximal 1-code C. Then $\theta_2 = -1$, the distance graph Γ_3 is strongly regular with parameters $(k_3, \lambda_3, \mu_3) = ((p + 1)(a + 1), p, a + 1)$, and C is a perfect 1-code. If the distance graph Γ_2 is strongly regular as well, then $a = c + 1$, $\theta_1 = c + p + 1$, $\theta_3 = -c - 1$ and the parameters of the corresponding strongly regular graph are*

$$(k_2, \lambda_2, \mu_2) = \big((c + 1)p(p + 1), (c + 1)(p^2 - p + 2) - p, (c + 1)p(p - 1)\big).$$

#	intersection array	p	status	#	intersection array	p	status
A1	$\{5,4,2;1,1,4\}$	4	! Sylvester graph [1, 13.1A]	B1	$\{6,4,2;1,2,3\}$	1	! $H(3,3)$, family (3)
A2	$\{35,24,8;1,6,28\}$	4	open, family (1), family (2)	B2	$\{12,10,2;1,2,8\}$	2	open
A3	$\{44,30,5;1,3,40\}$	10	does not exist by [4]	B3	$\{12,10,3;1,3,8\}$	2	! Doro graph [1, 12.1], family (4)
A4	$\{48,35,9;1,7,40\}$	5	open, family (2)	B4	$\{18,10,4;1,4,9\}$	1	! $J(9,3)$
A5	$\{49,36,8;1,6,42\}$	6	open	B5	$\{24,21,3;1,3,18\}$	3	open
A6	$\{54,40,7;1,5,48\}$	8	open	B6	$\{25,24,3;1,3,20\}$	4	open
A7	$\{63,48,10;1,8,54\}$	6	open, family (2)	B7	$\{30,28,2;1,2,24\}$	4	open
A8	$\{80,63,11;1,9,70\}$	7	open, family (2)	B8	$\{40,33,3;1,3,30\}$	3	open
A9	$\{99,80,12;1,10,88\}$	8	open, family (2)	B9	$\{40,33,8;1,8,30\}$	3	open, family (3)
A10	$\{119,96,18;1,16,102\}$	6	open, family (1)	B10	$\{50,44,5;1,5,40\}$	4	open
				B11	$\{60,52,10;1,10,48\}$	4	open, family (4)
				B12	$\{65,56,5;1,5,52\}$	4	open
				B13	$\{72,70,8;1,8,63\}$	7	open
				B14	$\{75,64,8;1,8,60\}$	4	open
				B15	$\{80,63,12;1,12,60\}$	3	open

Table 2: Intersection arrays corresponding to (9) and (10) from [1, pp. 425–431].

Proof. Since $k = b_2 + c_3 - 1$, Γ has eigenvalue -1 and Γ_3 is strongly regular with the desired parameters according to [1, Prop. 4.2.17]. As

$$|V\Gamma| = 1 + a(p+1) + ap(p+1) + (a+1)(p+1) = (1 + a(p+1))(p+2) = (1+k)|C| ,$$

the sphere packing bound is achieved, so C is a perfect 1-code. The distance graph Γ_2 is strongly regular if and only if $c_3(a_3 + a_2 - a_1) = b_1 a_2$ by [1, Prop. 4.2.17], in which case $a = c+1$, the intersection array of Γ is

$$\{(c+1)(p+1), cp, c+2; 1, c, (c+1)p\}$$

and the rest of the statement follows directly. □

Let Γ be a distance-regular graph with intersection array (9). We note that $a = c+1$ for all open cases A2–A10, but not for A1, i.e., the Sylvester graph. If we assume $a = c+1$ and that Γ additionally has

- a vanishing Krein parameter, then this is precisely q^3_{11}, which implies Γ is Q-polynomial for the natural ordering [5]. In this case $c = (p^2-4)/2$ and by integrality of c also $p = 2r$ for $r \in \mathbb{N}\backslash\{1\}$, which gives us precisely the intersection array (1).
- $p = c-2$, we obtain the feasible family (2). The Krein bound $q^3_{11} \geq 0$ is equivalent to $c \geq 6$. This family covers A2, A4, A7, A8 and A9 in Table 2.

Let Γ be a distance-regular graph with intersection array (10). If we assume that Γ has

- $q^3_{11} = 0$, which implies Γ is Q-polynomial for the natural ordering [5], then we have $p = 2r-1$, $a = t(2r+1)$ and $c = r(r+t)$ for some $r, t \in \mathbb{N}$, so its intersection array is

$$\{2rt(2r+1), (t(2r+1)+1)(2r-1), r(r+t); 1, r(r+t), t(4r^2-1)\}, \quad t \geq r \geq 1 . \tag{11}$$

  The condition $t \geq r$ is implied by the nonnegativity of q^3_{33}. Note that for $r = 2$ and $t = 4$ we obtain B15. If Γ is additionally formally self-dual, this implies $t = r$, which gives us precisely the intersection array (3).

- $\theta_2 = 0$, then $a = c(p+2)/(p+1)$, which implies $c = r(p+1)$ for some $r \in \mathbb{N}$ and furthermore $a = r(p+2)$. Integrality of m_1 requires $2r+1 \mid p(p+1)(p+2)$. Setting $p = 2r-1$ gives us precisely the intersection array (3), while setting $p = 2r$ gives us the feasible family (4).

4 Intersection numbers and eigenmatrices

Let Γ be a distance-regular graph and $u, v, w \in V\Gamma$. The relations on the triple intersection numbers $[i\ j\ h]$ $(0 \le i, j, h \le d)$ corresponding to (u, v, w) in the system (7) can be interpreted in terms of distance distributions corresponding to two vertices (the sizes of parts are the intersection numbers). In this section, we will specify the intersection numbers and eigenvalues of Γ for (1), (3) and (11), which will be then used in Sections 5 and 6 to compute the values $[i\ j\ h]$.

Let us first define the matrices $H_h := \left(p^h_{ij}\right)_{i,j=0}^{d}$ $(0 \le h \le d)$. For $d = 3$, we then have:

$$H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & a_1 & b_1 & 0 \\ 0 & b_1 & p^1_{22} & p^1_{32} \\ 0 & 0 & p^1_{23} & p^1_{33} \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & c_2 & a_2 & b_2 \\ 1 & a_2 & p^2_{22} & p^2_{32} \\ 0 & b_2 & p^2_{23} & p^2_{33} \end{pmatrix}, \quad H_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & c_3 & a_3 \\ 0 & c_3 & p^3_{22} & p^3_{32} \\ 1 & a_3 & p^3_{23} & p^3_{33} \end{pmatrix}.$$

Let us suppose Γ has intersection array (1), resp. (3). Then the intersection numbers p^h_{ij} $(0 \le i, j, h \le 3)$ are recorded in the matrices $\mathrm{diag}(k_0, k_1, k_2, k_3)$, H_1, H_2 and H_3. We also compute the entries of the eigenmatrix P and dual eigenmatrix Q using the sequence of cosines [3, 13.2]. (Recall that we are using the natural ordering of the minimal idempotents.) For (1), resp. (3), the matrices H_1, H_2, H_3 and P, Q are equal to

$$H_1: \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2(r^2+r-1) & 4r(r^2-1) & 0 \\ 0 & 4r(r^2-1) & 2r(2r-1)(2r^2-1) & 4r^3 \\ 0 & 0 & 4r^3 & 2r^2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & r(2r-1) & (2r-1)s & 0 \\ 0 & (2r-1)s & (2r-1)^2 s & (2r-1)s \\ 0 & 0 & (2r-1)s & s \end{pmatrix};$$

$$H_2: \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2(r^2-1) & (2r-1)(2r^2-1) & 2r^2 \\ 1 & (2r-1)(2r^2-1) & 2(4r^4-2r^3-1) & 2r^2(2r-1) \\ 0 & 2r^2 & 2r^2(2r-1) & 2r^2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2r^2 & 2r^2(2r-1) & 2r^2 \\ 1 & 2r^2(2r-1) & r(4r^2+3)(2r-1)-2 & 2r(s-2r) \\ 0 & 2r^2 & 2r(s-2r) & 2r^2 \end{pmatrix};$$

$$H_3: \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2r(2r^2-1) & 2r^2-1 \\ 0 & 2r(2r^2-1) & 2r(2r-1)(2r^2-1) & 2r(2r^2-1) \\ 1 & 2r^2-1 & 2r(2r^2-1) & 2r \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & r(4r^2-1) & r(2r+1) \\ 0 & r(4r^2-1) & (s-2r)(4r^2-1) & r(4r^2-1) \\ 1 & r(2r+1) & r(4r^2-1) & 2r-1 \end{pmatrix};$$

$$P = Q: \begin{pmatrix} 1 & (2r^2-1)(2r+1) & 2r(2r+1)(2r^2-1) & 2r^2(2r+1) \\ 1 & 2r^2+2r-1 & -2r & -2r^2 \\ 1 & -1 & -2r & 2r \\ 1 & -2r^2+1 & 4r^2-2 & -2r^2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2r^2(2r+1) & (4r^2-1)s & 2rs \\ 1 & r(2r+1) & 0 & -s \\ 1 & 0 & -2r-1 & 2r \\ 1 & -r(2r+1) & 4r^2-1 & -2r^2+r \end{pmatrix};$$

where $s = 2r^2 + r + 1$. As $P = Q$, the graph Γ is formally self-dual. In particular, this implies $p^h_{ij} = q^h_{ij}$ for $0 \le i, j, h \le 3$. For (1) the distance graphs Γ_2 and Γ_3 are strongly regular with eigenvalues $2r(2r+1)(2r^2-1), 4r^2-2, -2r$ and $2r^2(2r+1), 2r, -2r^2$, respectively (see the last two columns of P).

Let us now suppose Γ has intersection array (11). In Sections 5 and 6, we will need the following data:

$$\begin{array}{rclrclrclrcl} a_1 &=& 2r(t-1)+t, & a_2 &=& 2r^2(2t-1), & a_3 &=& t(2r+1), & c_3 = p^3_{23} &=& t(4r^2-1), \\ p^1_{33} &=& s, & & & b_2 = c_2 = p^2_{33} &=& r(r+t), & & p^3_{33} &=& 2r-1, \end{array}$$

$$P = \begin{pmatrix} 1 & 2rt(2r+1) & 2st(4r^2-1)/(r+t) & 2rs \\ 1 & t(2r+1) & 0 & -s \\ 1 & t-r & -2t-1 & r+t \\ 1 & -r(2r+1) & 4r^2-1 & -r(2r-1) \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & 2r & 2rst(4r^2-1) & 2st \\ 1 & 1 & -s(r-t)(2r-1) & -s \\ 1 & 0 & -r(r+t)(2t+1) & r+t \\ 1 & -1 & t(r+t)(4r^2-1) & -t(2r-1) \end{pmatrix} \cdot \operatorname{diag} \begin{pmatrix} 1 \\ (2rs-r-t)/(r+t) \\ (2r+1)/((r+t)(2r^2+t)) \\ (2rs-r-t)/((r+t)(2r^2+t)) \end{pmatrix},$$

where $s = 2rt+t+1$. Unlike in the previous cases, Γ is not formally self-dual in general. It is, however, Q-polynomial with $q^3_{11} = q^1_{13} = q^1_{31} = 0$.

5 Triple intersection numbers

Let Γ be a distance-regular graph with intersection array (1) or (11). We show that there exists a maximal 1-code in the graph Γ (Theorem 5.2) and then show that it is also locally regular and last subconstituent perfect. Note that the results in this section also apply to the family (3) that is a subfamily of (11). We start with a preliminary result that will be crucial in the proof of Theorem 5.2.

Lemma 5.1. *Let Γ be as distance-regular graph with intersection array (1) or (11) and u, v and w its vertices such that $\partial(u,v) = \partial(u,w) = \partial(v,w) = 3$. Then the following equations hold:*

$$\begin{bmatrix} u & v & w \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 3 & 2 \end{bmatrix} = 0, \tag{12}$$

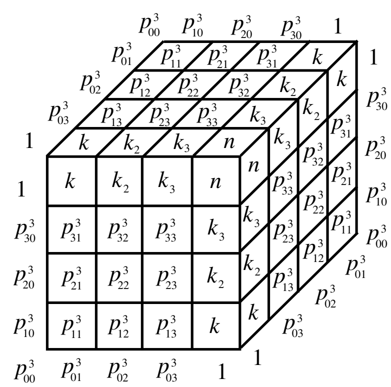
$$\begin{bmatrix} u & v & w \\ 3 & 3 & 3 \end{bmatrix} = p^3_{33} - 1. \tag{13}$$

Proof. Set $\alpha = \begin{bmatrix} u & v & w \\ 1 & 2 & 2 \end{bmatrix}$, $\beta = \begin{bmatrix} u & v & w \\ 2 & 1 & 2 \end{bmatrix}$, $\gamma = \begin{bmatrix} u & v & w \\ 2 & 2 & 1 \end{bmatrix}$, and $\delta = \begin{bmatrix} u & v & w \\ 3 & 3 & 3 \end{bmatrix}$. The system of equations (7) has a four-parametrical solution that can be expressed with α, β, γ and δ that are integral and nonnegative.

The general solution to the system is shown in Table 3. The table is split in three slices, each of which contains all the variables $\begin{bmatrix} u & v & w \\ i & j & h \end{bmatrix}$ for a fixed i. By arranging the slices from bottom to top, we get a $3 \times 3 \times 3$ cube with the variables $\begin{bmatrix} u & v & w \\ i & j & h \end{bmatrix}$ $(1 \le i, j, h \le 3)$ as its cells (see (6) and (7)).

Case 1. For the intersection array (1), we now obtain from Table 3:

$$\begin{array}{rcl} 2r^2 - 1 - \begin{bmatrix} u & v & w \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 1 & 3 & 2 \end{bmatrix} &=& 4r^3 - 2r - \alpha, \\ 2r^2 - 1 - \begin{bmatrix} u & v & w \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 2 & 1 & 3 \end{bmatrix} &=& 4r^3 - 2r - \beta, \\ 2r^2 - 1 - \begin{bmatrix} u & v & w \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 2 & 1 \end{bmatrix} &=& 4r^3 - 2r - \gamma, \\ \alpha + \beta + \gamma + \delta + \begin{bmatrix} u & v & w \\ 2 & 2 & 2 \end{bmatrix} &=& 8r^4 - 6r^2 + 2r, \\ \gamma - \begin{bmatrix} u & v & w \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 2 & 3 & 3 \end{bmatrix} &=& 4r^3 - 2r^2 - \alpha - \delta, \\ \alpha - \begin{bmatrix} u & v & w \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 2 & 3 \end{bmatrix} &=& 4r^3 - 2r^2 - \beta - \delta, \\ \beta - \begin{bmatrix} u & v & w \\ 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 3 & 2 \end{bmatrix} &=& 4r^3 - 2r^2 - \gamma - \delta. \end{array} \tag{14}$$



$[1\ 1\ 1]$	$=$	$0 \quad \Delta$	$[1\ 2\ 1]$	$=$	$0 \quad \Delta$	$[1\ 3\ 1]$	$=$	$0 \quad \Delta$
$[1\ 1\ 2]$	$=$	$0 \quad \Delta$	$[1\ 2\ 2]$	$=$	α	$[1\ 3\ 2]$	$=$	$c_3-\alpha$
$[1\ 1\ 3]$	$=$	$0 \quad \Delta$	$[1\ 2\ 3]$	$=$	$c_3-\alpha$	$[1\ 3\ 3]$	$=$	$a_3-c_3+\alpha$
$[2\ 1\ 1]$	$=$	$0 \quad \Delta$	$[2\ 2\ 1]$	$=$	γ	$[2\ 3\ 1]$	$=$	$c_3-\gamma$
$[2\ 1\ 2]$	$=$	β	$[2\ 2\ 2]$	$=$	$p^3_{22}-\gamma-[2\ 2\ 3]$	$[2\ 3\ 2]$	$=$	$p^3_{23}-c_3+\gamma-[2\ 3\ 3]$
$[2\ 1\ 3]$	$=$	$c_3-\beta$	$[2\ 2\ 3]$	$=$	$p^3_{23}-c_3+\alpha-[3\ 2\ 3]$	$[2\ 3\ 3]$	$=$	$p^3_{33}+c_3-a_3-1-\alpha-\delta$
$[3\ 1\ 1]$	$=$	$0 \quad \Delta$	$[3\ 2\ 1]$	$=$	$c_3-\gamma$	$[3\ 3\ 1]$	$=$	$a_3-c_3+\gamma$
$[3\ 1\ 2]$	$=$	$c_3-\beta$	$[3\ 2\ 2]$	$=$	$p^3_{32}-c_3+\beta-[3\ 3\ 2]$	$[3\ 3\ 2]$	$=$	$p^3_{33}+c_3-a_3-1-\gamma-\delta$
$[3\ 1\ 3]$	$=$	$a_3-c_3+\beta$	$[3\ 2\ 3]$	$=$	$p^3_{33}+c_3-a_3-1-\beta-\delta$	$[3\ 3\ 3]$	$=$	δ

Table 3: The general solutions for (7) in the setting of Lemma 5.1. The values marked with Δ are zero due to the triangle inequality.

Since $q^3_{11}=q^1_{13}=q^1_{31}=0$, Theorem 2.1 and (14) give the system of three equations

$$\begin{aligned}
(1-r)\alpha + r\beta + r\gamma + (1-r)\delta &= 4r^4+2r^3-6r^2+2r, && (15)\\
r\alpha + (1-r)\beta + r\gamma + (1-r)\delta &= 4r^4+2r^3-6r^2+2r, && (16)\\
r\alpha + r\beta + (1-r)\gamma + (1-r)\delta &= 4r^4+2r^3-6r^2+2r. && (17)
\end{aligned}$$

By solving (15–17) for α, β and γ, we obtain

$$\alpha=\beta=\gamma=4r^3-2r^2-\frac{2r(2r-1)-\delta(r-1)}{r+1}. \tag{18}$$

Plugging α,β,γ expressed by δ and r in equation (18) into the last three equations of (14) gives us

$$\begin{aligned}
\begin{bmatrix} u & v & w\\ 2 & 3 & 3\end{bmatrix} = \begin{bmatrix} u & v & w\\ 3 & 2 & 3\end{bmatrix} = \begin{bmatrix} u & v & w\\ 3 & 3 & 2\end{bmatrix} &= \frac{2r(2r-1-\delta)}{r+1},\\
\begin{bmatrix} u & v & w\\ 1 & 3 & 3\end{bmatrix} = \begin{bmatrix} u & v & w\\ 3 & 1 & 3\end{bmatrix} = \begin{bmatrix} u & v & w\\ 3 & 3 & 1\end{bmatrix} &= \frac{(r-1)(-2r+1+\delta)}{r+1},
\end{aligned}$$

implying $\delta=2r-1=p^3_{33}-1$. Therefore, the system of equations (14–17) has a single solution that agrees with the statement of the lemma.

Case 2. For the intersection array (11), we obtain

$$\begin{aligned}
t(2r+1)-\begin{bmatrix} u & v & w\\ 1 & 3 & 3\end{bmatrix} = \begin{bmatrix} u & v & w\\ 1 & 2 & 3\end{bmatrix} = \begin{bmatrix} u & v & w\\ 1 & 3 & 2\end{bmatrix} &= t(4r^2-1)-\alpha,\\
t(2r+1)-\begin{bmatrix} u & v & w\\ 3 & 1 & 3\end{bmatrix} = \begin{bmatrix} u & v & w\\ 3 & 1 & 2\end{bmatrix} = \begin{bmatrix} u & v & w\\ 2 & 1 & 3\end{bmatrix} &= t(4r^2-1)-\beta,\\
t(2r+1)-\begin{bmatrix} u & v & w\\ 3 & 3 & 1\end{bmatrix} = \begin{bmatrix} u & v & w\\ 2 & 3 & 1\end{bmatrix} = \begin{bmatrix} u & v & w\\ 3 & 2 & 1\end{bmatrix} &= t(4r^2-1)-\gamma,\\
\begin{bmatrix} u & v & w\\ 2 & 3 & 3\end{bmatrix} &= 2(r-1)(2rt+t+1)-\alpha-\delta,\\
\begin{bmatrix} u & v & w\\ 3 & 2 & 3\end{bmatrix} &= 2(r-1)(2rt+t+1)-\beta-\delta,\\
\begin{bmatrix} u & v & w\\ 3 & 3 & 2\end{bmatrix} &= 2(r-1)(2rt+t+1)-\gamma-\delta.
\end{aligned} \tag{19}$$

As the Krein parameters q^3_{11}, q^1_{13} and q^1_{31} of Γ vanish, we derive additional equations by Theorem 2.1 and (19). Since $Q_{21}=0$, the triple intersection numbers $[i\ j\ h]$ where at least two of i,j,h are 2 are always multiplied by zero in the equation from Theorem 2.1, so we omit them in (19). The obtained equations are

$$\begin{aligned}
-(r+1)\alpha + r\beta + r\gamma - r\delta &= 2(r-1)(2r^2t-rt-r-t), && (20)\\
r\alpha - (r+1)\beta + r\gamma - r\delta &= 2(r-1)(2r^2t-rt-r-t), && (21)\\
r\alpha + r\beta - (r+1)\gamma - r\delta &= 2(r-1)(2r^2t-rt-r-t). && (22)
\end{aligned}$$

By solving (20–22) for α, β, γ, we obtain

$$\alpha = \beta = \gamma = 4r^2t - 2rt - 2r - 2t + \frac{r\delta}{r-1}. \tag{23}$$

Plugging α, β, γ expressed in terms of δ and r in equation (23) into the last three equations of (19) gives us

$$\begin{aligned} \begin{bmatrix} u & v & w \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 3 & 2 \end{bmatrix} &= \frac{(2r-1)(2r-2-\delta)}{r-1}, \\ \begin{bmatrix} u & v & w \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 3 & 1 \end{bmatrix} &= \frac{r(-2r+2+\delta)}{r-1}, \end{aligned}$$

implying $\delta = 2r - 2 = p^3_{33} - 1$. Therefore, the system of equations (19–22) has a single solution that agrees with the statement of the lemma. □

Theorem 5.2. *Let Γ be a distance-regular graph with intersection array (1) or (11). Then, for any two vertices u, v of Γ with $\partial(u, v) = 3$, the graph Γ contains a unique maximal 1-code C such that $u, v \in C$, i.e. any 1-code in Γ containing u and v is a subset of C.*

Proof. From the equations (12) of Lemma 5.1 we see that choosing three vertices u, v, w in the graph Γ at distance 3 from each other gives us no vertices at distance 3 from u and v, and at distance 1 or 2 from w. Instead, the equation (13) tells us that there are $p^3_{33} - 1$ vertices at distance 3 from u, v and w. Since substituting w with each of the vertices from $\left\{ \begin{smallmatrix} u & v \\ 3 & 3 \end{smallmatrix} \right\}$ will give us the same equations, it means that all p^3_{33} vertices from $\left\{ \begin{smallmatrix} u & v \\ 3 & 3 \end{smallmatrix} \right\}$ are pairwise at distance 3 from each other. The set $C := \{u, v\} \cup \left\{ \begin{smallmatrix} u & v \\ 3 & 3 \end{smallmatrix} \right\}$ is therefore a maximal 1-code with $p^3_{33} + 2$ vertices, as defined in Section 3. Since C contains all the vertices at distance 3 from u and v, every 1-code of Γ containing both u and v is a subset of C. □

We conclude this section with a lemma and its corollary.

Lemma 5.3. *Let Γ be a distance-regular graph with intersection array (1) or (11) and C its maximal 1-code. Then C is both locally regular and last subconstituent perfect.*

Proof. Since we have $c_3 = p^3_{23} = a_3 p^3_{33}$ (see Section 4), C is locally regular by Proposition 3.2 and last subconstituent perfect by Proposition 3.3. □

Corollary 5.4. *Let Γ be a distance-regular graph with intersection array (1) or (11) and u', v, w vertices of Γ such that $\partial(u', v) = 1$, $\partial(u', w) = 2$ and $\partial(v, w) = 3$. Then we have $\left[\begin{smallmatrix} u' & v & w \\ 3 & 3 & 3 \end{smallmatrix} \right] = 1$.*

Proof. By Theorem 5.2, a maximal 1-code C such that $v, w \in C$ is uniquely defined by the vertices v, w at distance 3. The result then follows by Lemma 5.3 and Proposition 3.2. □

6 Nonexistence

We proceed in a manner similar to what we did in the proof of Lemma 5.1, only this time we choose three vertices u', v, w at pairwise distances 1, 2 and 3. The general solution of the system we obtain in this case is shown in Table 4. Note that $\left[\begin{smallmatrix} u' & v & w \\ 3 & 3 & 3 \end{smallmatrix} \right] = 1$ holds by Corollary 5.4.

Theorem 6.1. *A distance-regular graph with intersection array (1) does not exist. In particular, there is no distance-regular graph with intersection array A2 or A10, i.e.,*

$$\{35, 24, 8; 1, 6, 28\}, \quad \text{or} \quad \{119, 96, 18; 1, 16, 102\} \ .$$

Proof. Let Γ be a distance-regular graph with intersection array (1) and u', v and w' its vertices such that $\partial(u', v) = 1$, $\partial(u', w) = 2$ and $\partial(v, w) = 3$. Set $\alpha' = \begin{bmatrix} u' & v & w \\ 1 & 1 & 2 \end{bmatrix}$ and $\beta' = \begin{bmatrix} u' & v & w \\ 2 & 2 & 1 \end{bmatrix}$. The intersection numbers from Table 4 can be expressed in terms of r (see Section 4), giving:

$$\begin{aligned}
-2r + \alpha' + \begin{bmatrix} u' & v & w \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 1 & 2 & 1 \end{bmatrix} &= 2r^2 - 2, \\
2r - 1 + \begin{bmatrix} u' & v & w \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 2 & 2 & 3 \end{bmatrix} &= 4r^3 - 2r^2 - \alpha', \\
\begin{bmatrix} u' & v & w \\ 2 & 1 & 2 \end{bmatrix} &= 4r^3 - 2r - 1 - \alpha', \\
\begin{bmatrix} u' & v & w \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 2 & 1 & 3 \end{bmatrix} &= -2r + 1 + \alpha', \\
1 + \begin{bmatrix} u' & v & w \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 3 & 2 & 1 \end{bmatrix} &= 4r^3 - 2r^2 - 2r + 2 - \beta', \\
\begin{bmatrix} u' & v & w \\ 2 & 2 & 2 \end{bmatrix} &= 8r^4 - 8r^3 - 2r^2 + 2r + \alpha' - \beta', \\
2r^2 - 1 - \begin{bmatrix} u' & v & w \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 3 & 3 & 1 \end{bmatrix} &= -4r^3 + 4r^2 + 2r - 2 + \beta', \\
-\beta' + \begin{bmatrix} u' & v & w \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 3 & 2 & 3 \end{bmatrix} &= 2r^2 - 1, \\
-\beta' + \begin{bmatrix} u' & v & w \\ 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 2 & 3 & 3 \end{bmatrix} &= 2r - 1.
\end{aligned} \tag{24}$$

Since $q_{11}^3 = q_{13}^1 = 0$, Theorem 2.1 and (24) give the system of equations

$$-r^2\alpha' + (r-1)^2\beta' = 4r^5 - 14r^4 + 10r^3 + 2r^2 - 4r + 1, \tag{25}$$

$$-r\alpha' + r\beta' = 4r^4 - 6r^3 - 2r^2 + 4r - 1. \tag{26}$$

By solving (25–26) for α' and β', we obtain

$$\alpha' = 2r^2 + r - 3 + \frac{1}{r}, \tag{27}$$

$$\beta' = (4r^2 - 1)(r - 1). \tag{28}$$

It follows from the equation (27) and the integrality of α' and r that the system of equations (24–26) does not have any solutions for $r > 1$. We must then conclude that the choice of the vertices u', v, w of Γ is not possible. This contradicts the fact that $p_{23}^1 = 4r^3 > 0$. Therefore, the graph Γ does not exist. $\square$

The above result also shows the nonexistence of the first member of the infinite family of formally self-dual graphs with m^3 vertices and intersection array: $\{7(m-1), 6(m-2), 4(m-4); 1, 6, 28\}$, $m \geq 6$, which is realized when m is a power of two by a bilinear forms graph, see Brouwer et al. [1, p. 425].

Lemma 6.2. *Let Γ be a distance-regular graph with intersection array (11) and u', v and w its vertices such that $\partial(u', v) = 1$, $\partial(u', w) = 2$ and $\partial(v, w) = 3$. Then, $\begin{bmatrix} u' & v & w \\ 1 & 1 & 3 \end{bmatrix} = t$ holds.*

Proof. Let α' and β' be defined as in the proof of Theorem 6.1. The intersection numbers from Table 4 can be

$[1\ 1\ 1] = 0$	Δ	$[1\ 2\ 1] = c_2$		$[1\ 3\ 1] = 0$	Δ
$[1\ 1\ 2] = \alpha'$		$[1\ 2\ 2] = a_2 - \alpha'$		$[1\ 3\ 2] = 0$	Δ
$[1\ 1\ 3] = a_1 - \alpha'$		$[1\ 2\ 3] = b_2 - a_1 - 1 + \alpha'$		$[1\ 3\ 3] = 0$	Δ
$[2\ 1\ 1] = 0$	Δ	$[2\ 2\ 1] = \beta'$		$[2\ 3\ 1] = a_2 - \beta'$	
$[2\ 1\ 2] = c_3 - 1 - \alpha'$		$[2\ 2\ 2] = p^1_{22} - \beta' - [2\ 2\ 3]$		$[2\ 3\ 2] = p^3_{32} - [3\ 3\ 2]$	
$[2\ 1\ 3] = a_3 - a_1 + \alpha'$		$[2\ 2\ 3] = p^2_{23} - p^3_{33} + a_1 - a_3 + 1 - \alpha'$		$[2\ 3\ 3] = p^3_{33} - 1$	
$[3\ 1\ 1] = 0$	Δ	$[3\ 2\ 1] = c_3 - c_2 - \beta'$		$[3\ 3\ 1] = a_3 - a_2 + \beta'$	
$[3\ 1\ 2] = 0$	Δ	$[3\ 2\ 2] = p^2_{32} - [3\ 3\ 2]$		$[3\ 3\ 2] = p^1_{33} - 1 - [3\ 3\ 1]$	
$[3\ 1\ 3] = 0$	Δ	$[3\ 2\ 3] = p^2_{33} - 1$		$[3\ 3\ 3] = 1$	

Table 4: The general solutions for (7) in the setting of the proofs of Theorem 6.1 and Lemma 6.2. The values marked with Δ are zero due to the triangle inequality.

expressed in terms of r and t (see Section 4), giving:

$$\begin{aligned}
\begin{bmatrix} u' & v & w \\ 1 & 1 & 3 \end{bmatrix} &= 2r(t-1) + t - \alpha', \\
1 + \begin{bmatrix} u' & v & w \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 1 & 2 & 1 \end{bmatrix} &= r(r+t), \\
(r+1)(1-r+t) + \begin{bmatrix} u' & v & w \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 2 & 1 & 3 \end{bmatrix} &= 2r + \alpha', \\
\begin{bmatrix} u' & v & w \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 2 & 3 & 1 \end{bmatrix} &= 2r^2(2t-1) - \beta', \\
r(r+t) - \begin{bmatrix} u' & v & w \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} u' & v & w \\ 3 & 2 & 1 \end{bmatrix} &= t(4r^2-1) - r(r+t) - \beta', \\
\begin{bmatrix} u' & v & w \\ 2 & 3 & 3 \end{bmatrix} &= 2r - 2.
\end{aligned} \tag{29}$$

As in Case 2 of Lemma 5.1, we omit the values of $[i\ j\ h]$ where at least two of i, j, h are 2. Since $q^3_{11} = q^1_{13} = 0$, Theorem 2.1 and (29) give the system of equations

$$-r\alpha' + (r+1)\beta' = 2r(r^2(2t-1) - t), \quad \text{and} \quad -(r+1)\alpha' + r\beta' = 2r(r(r-1)(2t-1) - t + 1).$$

By solving it for α' and β', we get $\alpha' = 2r(t-1)$, $\beta' = 2r(2rt - r - t)$ and $\begin{bmatrix} u' & v & w \\ 1 & 1 & 3 \end{bmatrix} = 2r(t-1) + t - \alpha' = t$. □

We now restrict our focus to the family of distance regular graphs with intersection array (3), which is a subfamily of (11) for $t = r$.

Lemma 6.3. *Let Γ be a distance-regular graph with intersection array (3) and u', v and w' its vertices, such that $\partial(u', v) = \partial(u', w') = \partial(v, w') = 1$. Then, we have either*

(a) $\begin{bmatrix} u' & v & w' \\ 1 & 1 & 1 \end{bmatrix} = 0$, $\begin{bmatrix} u' & v & w' \\ 2 & 3 & 3 \end{bmatrix} = 2r^2 - r + 3$ *and* $\begin{bmatrix} u' & v & w' \\ 3 & 3 & 3 \end{bmatrix} = 2r - 2$, *or*

(b) $\begin{bmatrix} u' & v & w' \\ 1 & 1 & 1 \end{bmatrix} = r$, $\begin{bmatrix} u' & v & w' \\ 2 & 3 & 3 \end{bmatrix} = 2r^2 + 4$ *and* $\begin{bmatrix} u' & v & w' \\ 3 & 3 & 3 \end{bmatrix} = r - 3$.

Proof. Set $\gamma' = \begin{bmatrix} u' & v & w' \\ 1 & 1 & 1 \end{bmatrix}$ and $\delta' = \begin{bmatrix} u' & v & w' \\ 3 & 3 & 3 \end{bmatrix}$. The system of equations (7) has a biparametrical solution that can be expressed with γ' and δ' that are integral and nonnegative. Again, we show the general solution of the system in Table 5.

$$\begin{aligned}
\begin{bmatrix} u' & v & w' \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} u' & v & w' \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} u' & v & w' \\ 2 & 1 & 1 \end{bmatrix} &= 2r^2 - r - 1 - \gamma', \\
\begin{bmatrix} u' & v & w' \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} u' & v & w' \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u' & v & w' \\ 3 & 3 & 2 \end{bmatrix} &= 2r^2 + r + 1 - \delta'.
\end{aligned} \tag{30}$$

Once again we use the vanishing of q^3_{11} (see Theorem 2.1) and (30). Since $Q_{21} = 0$, we again omit the values $[i\ j\ h]$ where at least two of i, j, h are 2. This time we obtain the equation $(r+1)\gamma' + r\delta' = 2r^2 - 2r$. Since γ' and δ'

$[1\ 1\ 1] = \gamma'$	$[1\ 2\ 1] = a_1 - 1 - \gamma'$	$[1\ 3\ 1] = 0$ Δ
$[1\ 1\ 2] = a_1 - 1 - \gamma'$	$[1\ 2\ 2] = b_1 - a_1 + 1 + \gamma'$	$[1\ 3\ 2] = 0$ Δ
$[1\ 1\ 3] = 0$ Δ	$[1\ 2\ 3] = 0$ Δ	$[1\ 3\ 3] = 0$ Δ
$[2\ 1\ 1] = a_1 - 1 - \gamma'$	$[2\ 2\ 1] = b_1 - a_1 + 1 + \gamma'$	$[2\ 3\ 1] = 0$ Δ
$[2\ 1\ 2] = b_1 - a_1 + 1 + \gamma'$	$[2\ 2\ 2] = p^1_{22} - [2\ 1\ 2] - [2\ 3\ 2]$	$[2\ 3\ 2] = p^1_{23} - p^1_{33} + \delta'$
$[2\ 1\ 3] = 0$ Δ	$[2\ 2\ 3] = p^1_{23} - p^1_{33} + \delta'$	$[2\ 3\ 3] = p^1_{33} - \delta'$
$[3\ 1\ 1] = 0$ Δ	$[3\ 2\ 1] = 0$ Δ	$[3\ 3\ 1] = 0$ Δ
$[3\ 1\ 2] = 0$ Δ	$[3\ 2\ 2] = p^1_{23} - p^1_{33} + \delta'$	$[3\ 3\ 2] = p^1_{33} - \delta'$
$[3\ 1\ 3] = 0$ Δ	$[3\ 2\ 3] = p^1_{33} - \delta'$	$[3\ 3\ 3] = \delta'$

Table 5: The general solutions for (7) in the setting of Lemma 6.3. The values marked with Δ are zero due to the triangle inequality.

are integral and nonnegative, we derive two solutions, one for $\gamma' = 0$ and $\delta' = 2r-2$, and another for $\gamma' = r$ and $\delta' = r-3$. Thus

$$\text{(a)} \qquad \delta' = \begin{bmatrix} u' & v & w' \\ 3 & 3 & 3 \end{bmatrix} = 2r-2 \quad \Rightarrow \quad \begin{aligned} \begin{bmatrix} u' & v & w' \\ 1 & 1 & 1 \end{bmatrix} &= \gamma' &&= 0, \\ \begin{bmatrix} u' & v & w' \\ 2 & 3 & 3 \end{bmatrix} &= 2r^2 + r + 1 - \delta' &&= 2r^2 - r + 3; \end{aligned}$$

$$\text{(b)} \qquad \delta' = \begin{bmatrix} u' & v & w' \\ 3 & 3 & 3 \end{bmatrix} = r-3 \quad \Rightarrow \quad \begin{aligned} \begin{bmatrix} u' & v & w' \\ 1 & 1 & 1 \end{bmatrix} &= \gamma' &&= r, \\ \begin{bmatrix} u' & v & w' \\ 2 & 3 & 3 \end{bmatrix} &= 2r^2 + r + 1 - \delta' &&= 2r^2 + 4. \end{aligned}$$

□

Lemma 6.4. *Let Γ be a distance-regular graph with intersection array (3) and u', v its adjacent vertices. Then there are exactly $x = \dfrac{r(2r-1)(3-r)}{r+1}$ vertices $w' \in \begin{Bmatrix} u' & v \\ 1 & 1 \end{Bmatrix}$ such that $\begin{bmatrix} u' & v & w' \\ 1 & 1 & 1 \end{bmatrix} = 0$.*

Proof. Let x be the number of vertices $w'_1 \in \begin{Bmatrix} u' & v \\ 1 & 1 \end{Bmatrix}$ such that $\begin{bmatrix} u' & v & w'_1 \\ 1 & 1 & 1 \end{bmatrix} = 0$, and therefore $\begin{bmatrix} u' & v & w'_1 \\ 2 & 3 & 3 \end{bmatrix} = 2r^2 - r + 3$. For the remaining $a_1 - x$ vertices $w'_2 \in \begin{Bmatrix} u' & v \\ 1 & 1 \end{Bmatrix}$, $\begin{bmatrix} u' & v & w'_2 \\ 1 & 1 & 1 \end{bmatrix} = r$ and $\begin{bmatrix} u' & v & w'_2 \\ 2 & 3 & 3 \end{bmatrix} = 2r^2 + 4$ hold. A two way count of the pairs (w', w) such that $w' \in \begin{Bmatrix} u' & v \\ 1 & 1 \end{Bmatrix}$, $w \in \begin{Bmatrix} u' & v \\ 2 & 3 \end{Bmatrix}$ and $\delta(w, w') = 3$ gives us the relation $\begin{bmatrix} u' & v & w'_1 \\ 2 & 3 & 3 \end{bmatrix} x + \begin{bmatrix} u' & v & w'_2 \\ 2 & 3 & 3 \end{bmatrix} (a_1 - x) = \begin{bmatrix} u' & v & w \\ 1 & 1 & 3 \end{bmatrix} p^1_{23}$, which has the desired x as its only solution. □

Theorem 6.5. *A distance-regular graph Γ with intersection array (3) and $r \geq 2$ does not exist. In particular, there is no distance-regular graph with intersection array B9, i.e., $\{40, 33, 8; 1, 8, 30\}$.*

Proof. We consider three cases.

Case $r = 2$. Let u' and v be two adjacent vertices of Γ, and $w' \in \begin{Bmatrix} u' & v \\ 1 & 1 \end{Bmatrix}$. By Lemma 6.3, $\begin{bmatrix} u' & v & w' \\ 3 & 3 & 3 \end{bmatrix} = 2r - 2 = 2$ must hold for all $a_1 = 6$ choices of w' (as the other option is $r - 3 = -1 < 0$ and therefore invalid). However, even this is not possible, since by Lemma 6.4, this is only true for $x = 2$ choices of w'. Hence the graph Γ does not exist.

Case $r = 3$. Let u' and v be two adjacent vertices of Γ and w' any of their $a_1 = 15$ common neighbours. By Lemma 6.4, we have $x = 0$, so $\begin{bmatrix} u' & v & w' \\ 1 & 1 & 1 \end{bmatrix} = r = 3$. Therefore, the graph induced on the common neighbours of u' and v has 15 vertices and valency 3. This contradicts the hand-shake lemma, so Γ does not exist.

Case $r > 3$. By Lemma 6.4, $x < 0$ for $r > 3$, so Γ does not exist. □

7 Final remarks

We conclude our study with some remarks and pose some questions on the topic of this article.

Looking at the known examples of distance-regular graphs of diameter 3 with maximal 1-codes, it becomes natural to ask whether more general results could be obtained. For example, is a maximal 1-code in a distance-regular graph of diameter 3 always last subconstituent perfect? Is it always locally regular if the graph is additionally primitive? Are there, besides the Sylvester graph, any other graphs Γ with intersection array (9) such that Γ_2 is not strongly regular? Can we find more feasible families of distance-regular graphs that cover more examples from Table 2?

In Brouwer et al. [1, pp. 425–431] there are 10 feasible intersection arrays that provide potential counterexamples to the first two questions (i.e., strict inequalities in Proposition 3.2 and Corollary 3.4):

(C1) $\{22, 16, 5; 1, 2, 20\}$, (C2) $\{35, 30, 3; 1, 2, 25\}$, (C3) $\{39, 30, 4; 1, 5, 36\}$, (C4) $\{44, 35, 3; 1, 4, 42\}$, (C5) $\{44, 36, 5; 1, 9, 40\}$,
(C6) $\{44, 42, 5; 1, 7, 40\}$, (C7) $\{49, 36, 5; 1, 4, 45\}$, (C8) $\{74, 63, 5; 1, 9, 70\}$, (C9) $\{80, 72, 9; 1, 12, 72\}$, (C10) $\{90, 78, 7; 1, 13, 84\}$.

The existence of a graph with an intersection array from the above list and a maximal 1-code would provide a negative answer to both questions. We have not found a feasible intersection array for which equality holds in only one of the inequalities.

Besides the general proof given in Sections 5 and 6, some proofs of nonexistence exist for certain special cases. For the case of a graph Γ with intersection array (1) and $r > 3$, an alternative approach due to K. Coolsaet and M. Urlep independently (private communication) can be taken, by showing that Γ is locally strongly regular with parameters $k' = 2(r^2 + r - 1)$ and $\lambda' = \mu' = 2r - 1$. The nonexistence then follows from a two way counting of edges between $\left\{ \begin{smallmatrix} u' & w' \\ 1 & 1 \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} u' & w' \\ 1 & 2 \end{smallmatrix} \right\}$, where $\partial(u', w') = 1$, which leads to a contradiction.

An alternative proof for the nonexistence of a graph Γ with intersection array (3) and $r = 2$ was given by M. Urlep and the second author. For two adjacent vertices u' and w' of Γ, we compare the two way count of edges between $\left\{ \begin{smallmatrix} u' & w' \\ 1 & 1 \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} u' & w' \\ 2 & 2 \end{smallmatrix} \right\}$ to the two way count of 2-paths starting and ending in $\left\{ \begin{smallmatrix} u' & w' \\ 1 & 1 \end{smallmatrix} \right\}$ with the midpoint in $\left\{ \begin{smallmatrix} u' & w' \\ 2 & 2 \end{smallmatrix} \right\}$, and obtain a contradiction.

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