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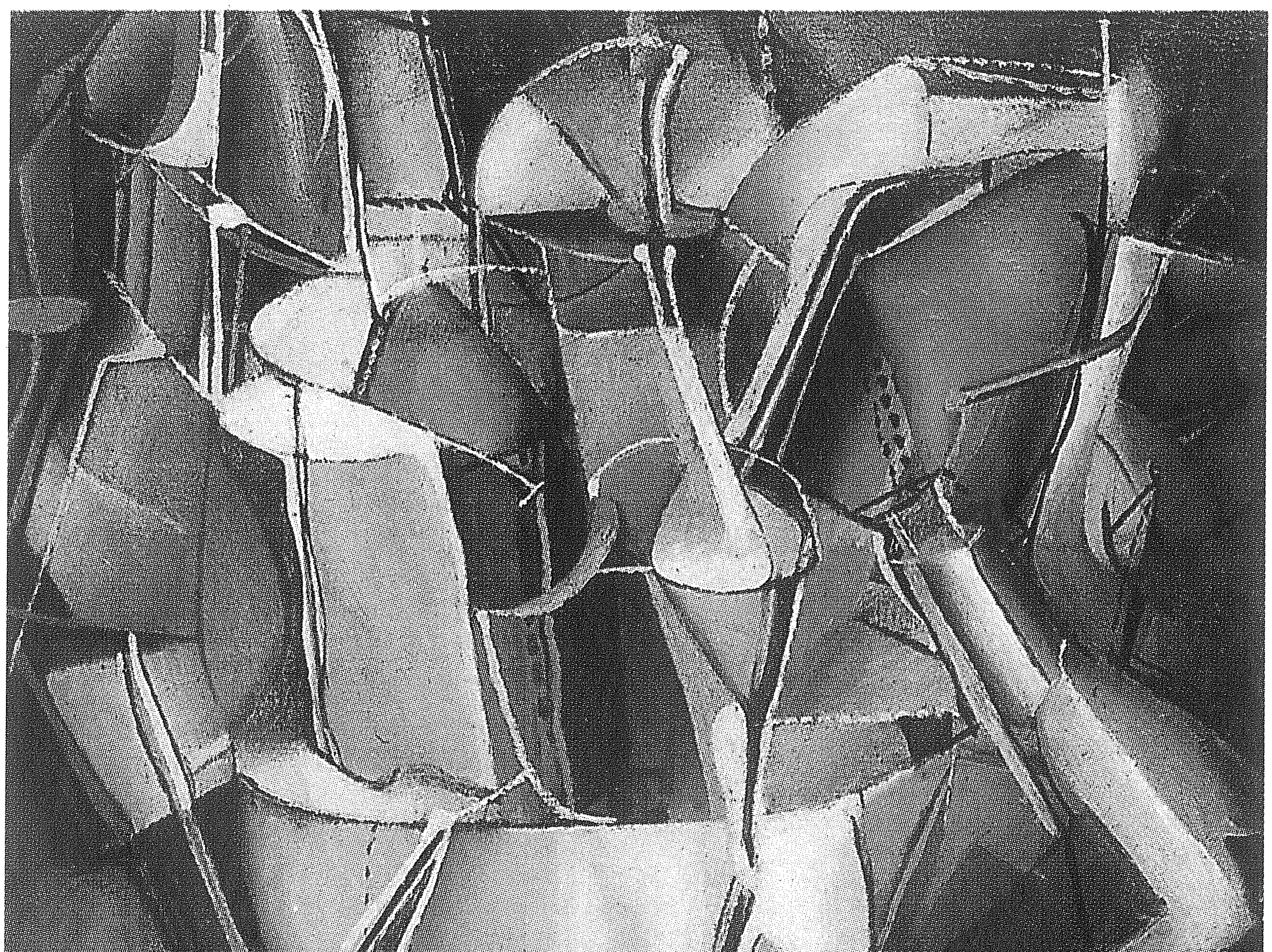
OBZORNIK ZA MATEMATIKO IN FIZIKO



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POSEBNA ŠTEVILKA

ob drugem srečanju iz linearne algebre
Bled 1999

To številko sta strokovno uredila Matjaž Omladič in Tomaž Košir, jezikovno pa jo je pregledala Anesa Bukič.

Na ovitku: Marcel Duchamp, *Pot od device do neveste*, 1912.

DRUGO SREČANJE IZ LINEARNE ALGEBRE BLED 1999

MATJAŽ OMLADIČ

V prispevku predstavimo srečanje iz linearne algebre, ki bo letos junija že drugič potekalo na Bledu. Podamo tudi povzetke vabljenih predavanj za letošnje srečanje.

THE SECOND MEETING IN LINEAR ALGEBRA BLED 1999

An introduction of the second meeting in Linear Algebra which will be held at Bled in June 1999 is given. Abstracts of the invited talks for the meeting are listed.

Linearna algebra je zanimiva veja matematike, ki zadnje čase vse bolj pridobiva na popularnosti. Razlog za to je njena široka uporabnost v različnih drugih vejah matematike in v drugih vedah. Prav zato se z njo ukvarjajo ljudje različnih zanimanj in različnih predznanj. K temu morda prispeva tudi dejstvo, da je mnoge probleme linearne algebre razmeroma lahko formulirati in so zato ti problemi (ne pa nujno tudi njihove rešitve) lahko razumljivi razmeroma širokemu krogu matematikov različnih profilov. Tudi znanstvenih srečanj s tega področja je veliko. Zato se kaže potreba po organizaciji manjših srečanj, na katerih bi se sestajali ljudje sorodnih interesov v linearni algebri. Blejsko srečanje namenja pozornost tistemu delu linearne algebre, kjer se stika algebra s funkcionalno analizo. Ta stik ima v razvoju slovenske matematike poseben pomen, saj ga je pri nas prvi gojil profesor Vidav, za njim pa še mnogi drugi slovenski matematiki. Tako se je v Sloveniji razvila prava šola, ki se ukvarja s tem področjem. Naši matematiki pa imajo na tem področju razvito tudi bogato mednarodno sodelovanje. Zato ni čudno, da so lahko pričeli z organizacijo teh srečanj pri nas. Prvo blejsko srečanje iz linearne algebre je potekalo od 20. 5. do 31. 5. 1996 v Hotelu Park na Bledu. Udeležili so se ga mnogi ugledni gosti iz tujine: Rajendra Bhatia (New Delhi, Indija), Paul A. Binding (Calgary, Kanada), Luzius Grunenfelder (Halifax, Kanada), John Holbrook (Guelph, Kanada), Ali A. Jafarian (New Haven, ZDA), Charles R. Johnson (Williamsburg, ZDA), Thomas J. Laffey (Dublin, Irska), Heinz Langer (Dunaj, Avstrija), Raphael Loewy (Haifa, Izrael), Martin Mathieu (Tübingen, ZRN), Branko Najman (Zagreb, Hrvaška), Denes Petz (Budimpešta, Madžarska), Steven J. Pierce (San Diego, ZDA), Vlastimil Ptak (Praga, Češka), Heydar Radjavi (Halifax, Kanada), Leiba Rodman (Williamsburg, ZDA), Peter Rosenthal (Toronto, Kanada) in Jaroslav Zemanek (Varšava, Poljska). Poleg teh so se srečanj udeležili tudi mnogi slovenski matematiki, tako ugledni profesorji kot tudi podiplomski študentje. Podobno kvalitetno udeležbo pričakujemo tudi letos, ko so svojo namero po sodelovanju poleg mnogih izmed zgoraj naštetih potrdili tudi: Man-Duen Choi (Toronto, Kanada), Raul Curto (Iowa City, ZDA), Alexander S. Fainshtein (Magnitogorsk, Rusija) in Wojtek Wojtinsky (Varšava, Poljska). Letošnje srečanje bo potekalo v Hotelu Park na Bledu od 1. 6. do 10. 6. 1999. Tudi tokrat nameravamo srečanje organizirati na podoben način kot pred tremi leti. Prve jutranje ure bodo namenjene vabljenim predavanjem, ki naj bi pomenila uvod v nadaljnje delo. Kasneje se bomo organizirali v delavnicah, kjer se bomo ukvarjali s konkretnimi odprtimi problemi. Nekoliko v nasprotju z navado na podobnih srečanjih bomo tako lahko posvetili več časa premišljevanju o problemih kot pa pripovedovanju formalnih predavanj. Ta način dela se je izkazal kot zelo ploden ob prvem srečanju, ko so izsledki dela mnogih delovnih skupin predstavljali osnovo za nekatere članke, ki so bili kasneje objavljeni v mednarodnih znanstvenih revijah. S tega zornega kota je bila morda najuspešnejša delovna skupina za študij nerazcepnosti in trikotljivosti operatorskih polgrup, ki je v toku srečanja v celoti rešila zanimiv problem, ki je že izšel kot članek sedmih avtorjev v ugledni ameriški reviji Journal of Functional Analysis.

POVZETKI VABLJENIH PREDAVANJ ZA LETOŠNJE SREČANJE ABSTRACTS OF INVITED TALKS FOR THE MEETING IN JUNE 1999

COMPLETELY POSITIVE MATRICES, GRAPHS WITH NO LONG ODD CYCLE AND GRAPHS WITH NO SHORT ODD CYCLE

AVRAM BERMAN

Department of Mathematics, Technion–Israel Institute of Technology, Haifa, Israel.

A matrix A is completely positive if it can be decomposed as $A = BB'$, where B is a (not necessarily square) elementwise nonnegative matrix. An obvious necessary condition for a symmetric nonnegative matrix to be completely positive is that it is positive semi definite. This condition is not sufficient. A sufficient condition for a symmetric nonnegative matrix to be completely positive is that its comparison matrix is positive semi definite. This condition, due to Drew, Johnson and Loewy, is not necessary. The sufficient condition is necessary if the graph of the matrix is triangle free (contains no short odd cycles). The necessary condition is sufficient if the graph contains no odd cycle of length greater than 4 (long odd cycle). We will discuss the relationship between these two results.

The smallest number of columns of B in the decomposition $A = BB'$ is called the cp-rank of A. We will discuss some results and conjectures on bounds for the cp-rank.

MULTIPARAMETER STURM-LIOUVILLE PROBLEMS WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS

TIRTHANKAR BHATTACHARYYA

(Joint work with Paul Binding and Karim Seddighi.)

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Linked equations

$$-y_i'' + q_i y_i = \sum_{j=1}^{n} \lambda_j r_{ij} y_i, \qquad i = 1, 2 \tag{1}$$

are studied on $[0,1]$ subject to boundary conditions of the form

$$y_i(0)\cos\alpha_i = y_i''(0)\sin\alpha_i \tag{2}$$

$$b_i y_i(1) - d_i y_i'(1) = \mathbf{e}_i^T \boldsymbol{\lambda}(c_i y_i'(1) - a_i y_i(1)), \tag{3}$$

where $[\mathbf{e}_1\ \mathbf{e}_2\ \cdots \mathbf{e}_n] \in \mathbb{R}^{n\times n}$ is an arbitrary matrix, $a_i d_i - b_i c_i \neq 0$ for all i and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ are parameters. Results are given on existence, location, asymptotics and perturbation of the eigenvalues λ_j and oscillation of the eigenfunctions y_i.

So far there seems to be no analysis of Sturm theory with $\boldsymbol{\lambda}$-dependent boundary conditions for more than one parameter, and it is our aim to start such a theory by considering a special case. We study (1) subject to (2) and (3). For $n = 2$, we study the eigencurves for (1) for each fixed i and we obtain expressions for the derivatives $\frac{d\lambda_2}{d\lambda_1}$ along the eigencurves, and certain asymptotics. We give the basic existence and uniqueness theorem for eigenvalues $\boldsymbol{\lambda}$ and we obtain an oscillation theorem which generalizes all the classical results. For $n > 2$, the theory of commuting self-adjoint operators on a Krein space arises naturally in the context. We refine the analysis to show existence and completeness under a general set-up and various definiteness conditions.

FREDHOLM FAMILIES OF OPERATORS GENERATING NILPOTENT LIE ALGEBRAS

ALEXANDER S. FAINSHTEIN
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In this communication we try to develop a Fredholm theory for families of operators in Banach spaces generating nilpotent Lie algebras. For the concept of Taylor joint spectrum and the polynomial spectral mapping theorem for such families of operators see the author's paper in J. Operator Theory, **29** (1993), 3–27.

Let E be a complex finite-dimensional nilpotent Lie algebra, a complex vector space X be a E-module and $\mathrm{Kos}(E,X)$ be the relative chain Koszul complex with the homology spaces $H_i(E,X)$. The complex $\mathrm{Kos}(E,X)$ is said to be Fredholm if the spaces $H_i(E,X)$ are finite-dimensional. Then the index of $\mathrm{Kos}(E,X)$ is defined as $\mathrm{ind}(E,X) = \sum_i (-1)^i \dim H_i(E,X)$.

Let $a = (a_1, ., a_n)$ be a family of bounded operators in a Banach space X generating nilpotent Lie algebra $E(a)$. We say a is Fredholm if $\mathrm{Kos}(E(a),X)$ is Fredholm and define its index by $\mathrm{ind}(a,X) = \mathrm{ind}(E(a),X)$. The essential spectrum $\sigma_e(a,X)$ of a is the set of $\lambda \in C^n$ such that $a - \lambda$ fails to be Fredholm. On the other hand, for given a we may take a nilpotent Lie algebra E with generators $e_1, ., e_n$ and define E-module structure on X by the Lie algebra homomorphism $\rho : E \to \mathcal{L}(X)$ with $\rho(e_i) = a_i$. If we define a to be Fredholm if $\mathrm{Kos}(E,X)$ is Fredholm we get an equivalent definition of the Fredholmness and the joint essential spectrum. The index however depends on the choice of E.

We prove the polynomial spectral mapping theorem for essential spectrum and analogues of known results on families of operators on tensor products. We also prove the following results on triviality of the index.

Let F and E be finite-dimensional nilpotent Lie algebras.

1. Let $\eta : F \to E$ be a Lie algebra epimorphism, X be an E-module (hence also an F-module). Complexes $\mathrm{Kos}(F, X)$ and $\mathrm{Kos}(E, X)$ are simultaneously Fredholm and if in this case η is not an isomorphism then $\mathrm{ind}(F, X) = 0$.

2. Let E be a proper Lie subalgebra of F and X be an F-module (hence also an E-module). If $\mathrm{Kos}(E, X)$ is Fredholm then $\mathrm{Kos}(F, X)$ is Fredholm and $\mathrm{ind}(F, X) = 0$.

I am deeply grateful to the organizers for the invitation and to the Open Society Institute in Ljubljana for the support allowing me to attend the 2nd Linear Algebra Workshop.

ASCENT AND DESCENT FOR COMMUTING ENDOMORPHISMS

LUZIUS GRUNENFELDER AND MATJAŽ OMLADIČ

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Let B be an algebra over a field and let M be a left B-module. If $a : M \to M$ is a B-endomorphism then $\ker a^i \subseteq \ker a^{i+1}$ and $a^i M \supseteq a^{i+1} M$ for every $i \geq 0$. The ascent of a is the least positive integer r for which $\ker a^r = \ker a^{r+1}$ and the descent is the least positive integer s for which $a^s M = a^{s+1} M$, if such integers exist and ∞ if they don't. If both the ascent r and the descent s of a are finite then $r = s$ and $M = \ker a^r \oplus a^r M$. This is Fitting's Lemma. It holds in particular for every $a \in \mathrm{End}_B(M)$ if the B-module M is both Artinian and Noetherian. More generally, we may say that a has the Fitting property if $M = K \oplus J$, where $K = \cup_i \ker a^i$ and $J = \cap_i a^i M$.

Here we use homological techniques involving the Koszul complex to define and explore the notion of ascent and descent, as well as a Fitting type decomposition of M, for finite sequences $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ of commuting endomorphisms of a B-module M. The approach works for any module M over a commutative ring A and any finite sequence of elements of A acting as endomorphisms, without specific reference to an B-module structure on M. However, in our context the Fitting decomposition is of course B-invariant. In general, we say that the n-tuple $\mathbf{a}$ has the Fitting property if $M = K \oplus J$ and $\langle \mathbf{a} \rangle J = J$, where $K = \cup_i \mathrm{Hom}_A(A/\langle \mathbf{a} \rangle^i, M)$, $J = \cap_i \langle \mathbf{a} \rangle^i M$ and $\langle \mathbf{a} \rangle$ is the ideal in A generated by the n-tuple $\mathbf{a}$. The main result is that $\mathbf{a}$ has finite ascent and finite descent if and only if $\mathbf{a}$ has the Fitting property and $\langle \mathbf{a} \rangle$ acts nilpotently on K. It turns out that ascent and descent are invariants of the ideal $\langle \mathbf{a} \rangle$ in A, in fact of the subspace generated by $\mathbf{a}$, i.e. they are independent of the choice of the finite generating set, even in the absence of the Fitting property.

Apart from their independent interest, the results presented have also been motivated by the question of how to study the spectral behaviour of a sequence of commuting bounded operators. Our approach provides a Fitting type decomposition for every point in the Taylor spectrum.

COMPLETELY RANK-NONINCREASING LINEAR MAPS

DON HADWIN

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We discuss operator-valued linear mappings on a linear space of operators that are completely rank-nonincreasing (in the sense of completely positive or completely bounded maps) . We conjecture that these are precisely the pointwise strong-operator limits of elementary maps. We show that partial positive results, combined with Voiculescu's theorem on approximate unitary equivalence lead to completely algebraic characterizations of approximate summands, compressions and skew-compressions of a representation of a C^*-algebra. We also use this concept to provide counterexamples to two conjectures of R. Curto and D. Herrero on joint similarity orbits of matrices.

SCHUR NORMS – COMPUTATION AND APPLICATION

JOHN HOLBROOK

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By the *Schur norm* $||M||_S$ of an $n \times n$ matrix M we mean its norm as a Schur multiplier on $(M_n(\mathbb{C}), ||\cdot||)$, where $||\cdot||$ denotes the operator norm. Thus

$$||M||_S = \max\{||M \circ X|| : ||X|| \leq 1\},$$

where $M \circ X$ is the Schur (elementwise) product of the matrices M and X. This norm is notoriously hard to compute (even for 2×2 matrices!); we review methods of finding $||M||_S$, some going back to Schur and some depending on *explicit Haagerup factorization* via

$$||M||_S = \min\{||X||_r ||Y||_c : XY = M\},$$

where $||\cdot||_r$ and $||\cdot||_c$ denote the max–row and max–column norms of their matrix arguments. We consider a number of applications. We may briefly recall applications discussed during LAW 96; for example, Schur norm computations shed light on multivariate von Neumann inequalities of the type

$$||p(C_1, C_2, C_3)|| \leq \text{Const} \cdot ||p||_\infty,$$

where the C_k are commuting contractions and p is analytic on the polydisc. Details may be found in [1]. Among more recent applications, we report on work with R. Bhatia (see [2]) concerning Fréchet derivatives of the power function: let $A \to A^r$ be the map that takes a positive definite matrix to its rth power, and let DA^r be the Fréchet derivative of this map. We show that $\|DA^r\| = \|rA^{r-1}\|$ precisely when r is *not* in the interval $(1/\sqrt{2}, 2)$. We also summarize work with F. Gilfeather (see [3]) on the Pedersen conjecture about commutators: for positive definite matrices A and B, and contraction C, and any matrix–monotone function f,

$$\|f(A)C - Cf(B)\| \le f(\|AC - CB\|).$$

References

[1] J. Holbrook, *Schur norms and the multivariate von Neumann inequality*, preprint.
[2] R. Bhatia and J. Holbrook, *Noncommutative Fréchet derivatives*, preprint.
[3] F. Gilfeather and J. Holbrook, *On the Pedersen conjecture*, preprint.

SOME NEW INVARIANTS RELATING TO THE SIMULTANEOUS SIMILARITY OF MATRICES

THOMAS J. LAFFEY

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Let $M_n(\mathbf{C})$, $M_n(\mathbf{C}[x])$ denote the rings of $n \times n$ matrices over $\mathbf{C}$ and the associated polynomial ring $\mathbf{C}[x]$, respectively, and let $GL(n, \mathbf{C})$, $GL(n, \mathbf{C}[x])$ denote, as usual, the groups of units of these rings. We say that a pair of elements $A(x)$, $B(x)$ in $M_n(\mathbf{C}[x])$ are *PS-equivalent* if there exists $P(x) \in GL(n, \mathbf{C}[x])$, $Q \in GL(n, \mathbf{C})$ with $B(x) = P(x)A(x)Q$.

Recall that $A(x)$ and $B(x)$ are *equivalent* if there exists $H(x)$, $K(x) \in GL(n, \mathbf{C}[x])$ with $B(x) = H(x)A(x)K(x)$, and that every element $A(x)$ is equivalent to a diagonal matrix $\operatorname{diag}(s_1(x), \ldots, s_r(x), 0, \ldots, 0)$ where r is the rank of $A(x)$, and where $s_1(x), \ldots, s_r(x)$ are canonically determined monic polynomials (called the *Smith invariants* or *invariant factors* of $A(x)$) with the property that $s_i(x)$ divides $s_{i+1}(x)$ for $i = 1, 2, \ldots, r-1$.

We show that if $\det A(x) \neq 0$, then $A(x)$ is PS-equivalent to an upper triangular matrix $S(x) = (s_{ij}(x))$ where the diagonal entries $s_{ii}(x)$ are the Smith invariants $s_i(x)$ and where for $j > i$, either $s_{ij}(x) = 0$ or $s_{ij}(x)$ is a monic polynomial having degree less than $\deg s_{jj}(x)$ and having $s_{ii}(x)$ as a proper divisor.

Such an $S(x)$ is called a near canonical form (NCF) of $A(x)$. The question of the PS-equivalence of $A(x)$, $B(x)$ can be reduced to that of the PS-equivalence of two NCFs $S(x)$, $T(x)$ with the same Smith invariants.

In the generic case in which the equation $\det A(x) = 0$ has distinct roots, we can assume the associated NCFs are

$$S(x) = \begin{bmatrix} I_{n-1} & u(x) \\ 0 & s_n(x) \end{bmatrix}, \quad T(x) = \begin{bmatrix} I_{n-1} & v(x) \\ 0 & s_n(x) \end{bmatrix},$$

and

$$u(x) = (u_1(x), \ldots, u_{n-1}(x))^T$$
$$v(x) = (v_1(x), \ldots, v_{n-1}(x))^T$$

where each nonzero $u_i(x), v_j(x)$ is a monic polynomial with $u_i(\lambda_1) = v_j(\lambda_1) = 0$, where λ_1 is a fixed root of $s_n(x) = 0$. Let $U = \operatorname{span}(u_1(x), \ldots, u_{n-1}(x))$, $V = \operatorname{span}(v_1(x), \ldots, v_{n-1}(x))$. Then if $A(x)$, $B(x)$ are PS-equivalent, $\dim U = \dim V$. Furthermore, if we write $s_n(x) = (x - \lambda_1) \cdots (x - \lambda_N)$ and for $0 \neq w(x) \in \mathbf{C}[x]$, define

$$\Phi(w(x)) = \max\{k \geq 0 \mid (x - \lambda_1) \cdots (x - \lambda_k) \text{ divides } w(x)\},$$

we can choose a basis of U inductively as follows:
Let $\hat{u}_1(x) \in U$ be a nonzero element $w(x)$ with $\Phi(w(x))$ maximal and having chosen $\hat{u}_1(x), \ldots, \hat{u}_k(x)$ and $k < \dim U$, we choose $\hat{u}_{k+1}(x)$ to be an element $z(x)$ of $U \backslash \operatorname{span}\{\hat{u}_1(x), \ldots, \hat{u}_k(x)\}$ with $\Phi(z(x))$ maximal. Choose a basis of V in the same way. Then if $A(x)$, $B(x)$ are PS-equivalent, $\Phi(\hat{u}_i(x)) = \Phi(\hat{v}_i(x))$ for all i.

An algorithm to determine PS-equivalence will be presented and applications given to the problem of determining the simultaneous similarity of two lists $(A_1, \ldots, A_t)$ and $(B_1, \ldots, B_t)$ of elements of $M_n(\mathbf{C})$.

This is joint work with J. A. Dias Da Silva.

RANK PRESERVERS ON SPACES OF SYMMETRIC MATRICES

RAPHAEL LOEWY

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Let $S_n(F)$ denote the set of all $n \times n$ symmetric matrices over the field F. Let k be a positive integer such that $k \leq n$. A linear operator T on $S_n(F)$ is said to be a rank-k preserver provided that it maps the set of all rank k matrices into itself.

Suppose that $k = 2r$ is an even integer. Beasley and Loewy showed that if F is algebraically closed of characteristic $\neq 2$, then any rank-k preserver on $S_n(F)$ must be a congruence map. They also showed that if $n \geq 2k = 4r$, any rank-k preserver on $S_n(\mathbb{R})$ must be a congruence map, possibly followed by negation. Following this and earlier results, the problem of characterizing rank-k preservers on $S_n(\mathbb{R})$ is still open if $k + 1 \leq n \leq 2k - 1$.

In this talk we describe an improvement of the Beasley-Loewy result for $S_n(\mathbb{R})$. It turns out that three types of subspaces are relevant to the investigation of rank-k preservers: (I) A subspace where each nonzero matrix has rank at least k. (II) A subspace where each nonzero matrix has rank equal to k. (III) A subspace where each matrix has rank at most k.

We consider those three types of subspaces. In particular, we obtain the following result: Suppose that F is an infinite field, $k = 2r$ and $2n > 5r+1$. Suppose that L is a subspace of $S_n(F)$ of type (III) such that

$$\dim L > \max\{2r^2 + 2r, \frac{1}{2}(2nr - 2n - r^2 + 3r + 4)\}.$$

Then L is decomposable. This is an analogue of a theorem of Atkinson and Lloyd, who considered a subspace of the space of all $m \times n$ matrices over F which is of type III.

Several open problems will also be mentioned.

STRONG PARTIAL ISOMETRIES

VLASTIMIL PTÁK

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The classical Toeplitz operators are defined as compressions to H^2 of the multiplication operator $M(\varphi)$ on L^2, φ being an L^∞ function. An operator $T : H^2 \to H^2$ is Toeplitz for a suitable φ if and only if it satisfies the relation $T = S^*TS$ where S is the (forward) shift operator on H^2. The function φ is defined uniquely by T. A Hankel operator corresponding to φ is the compression from H^2 to H^2_- of $M(\varphi)$, $P(H^2_-)M(\varphi)|H^2$. An operator $X : H^2 \to H^2_-$ is Hankel if and only if it satisfies the intertwining relation $XS = ZX$, Z being the backward shift on H^2_-; this relation does not define the corresponding φ uniquely. Sz. Nagy and Foias studied generalized Toeplitz operators on Hilbert spaces $X : \mathcal{H}_1 \to \mathcal{H}_2$ defined by the relation $X = T_2XT_1^*$, T_1 and T_2 being arbitrary contractions on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. They found a unique symbol Y constructed from the minimal isometric dilations U_1^* and U_2^* of T_1 and T_2 acting on $\mathcal{K}_1$ and $\mathcal{K}_2$. In this manner X appears as the compression $P(\mathcal{H}_2)Y|\mathcal{H}_1$ of Y. In Acta Sci. Math (Szeged) 52 (1988) P. Vrbová and the author introduced a Hankel operator corresponding to T_1 and T_2, imitating the classical case, replacing H^2_- by $\mathcal{K}_2 \ominus \mathcal{H}_2$. The generalized Hankel operator $H = P(\mathcal{K}_2 \ominus \mathcal{H}_2)Y|\mathcal{H}_1$ satisfies the intertwining relation $T_2H = HT_1^*$. To obtain an analogue of the Nehari theorem another condition has to be imposed, the so called R-boundedness – this condition is trivially satisfied in the classical case; its meaning clears up the general situation. Recently (Math. Bohemica **122** (1997)) the author described a wider class of Hankel type operators defined by the intertwining relation $T_2H = HT_1^*$ and other boundedness conditions.

One of the important tools in these investigations is the decomposition $\mathcal{R} = \mathcal{P} \oplus (\mathcal{R} \cap \mathcal{H}^\perp)$ where $\mathcal{P}$ is the closure of $P(\mathcal{R})\mathcal{H}$ and a related coisometry $W = (U^*|\mathcal{P})^*$. The particular case where $\mathcal{H}^\perp \subset \mathcal{R}$, in other words, $\mathcal{R} \cap \mathcal{H}^\perp = \mathcal{H}^\perp$ will be investigated, its motivation explained and the techniques for its solution described. The connection of this problem with strong partial isometries will be discussed. In a joint paper C. Mancera, P. Paúl, V. Pták, V. Vasyunin prove the following: given a contraction T on a Hilbert space such that T^n is a partial isometry for each $n \geq 0$ then $\mathcal{H}$ may be orthogonally decomposed into four T reducing subspaces such that the corresponding parts are (1) unitary (2) forward shift (3) backward shift, and (4) orthogonal sum of finite truncated shifts (some of the parts may be missing).

THE PERRON-FROBENIUS THEOREM REVISITED

HEYDAR RADJAVI
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The celebrated Perron-Frobenius Theorem makes several assertions about the form, spectrum, and fixed positive vectors of certain matrices with non-negative entries. It turns out that most of these assertions hold in a much more general setting, i.e., for a large class of multiplicative semigroups of non-negative matrices. These extensions will be discussed in this talk. There are no prerequisites except elementary linear algebra and elementary analysis.

A RESOLVENT CONDITION IMPLYING POWER BOUNDEDNESS.

JAROSLAV ZEMANEK
Institute of Mathematics, Polish academy of Sciences, Warsaw, Poland.

Analytic resolvent conditions will be related to the behaviour of the powers, their consecutive differences and Cesaro means. Some examples and characterizations of the extremal cases will motivate open questions for further research.

THE ROLE OF ASCENT AND DESCENT IN ANALYSIS.

JAROSLAV ZEMANEK
Institute of Mathematics, Polish academy of Sciences, Warsaw, Poland.

We intend to show the role of ascent, descent, and closedness of operator ranges in the ergodic behaviour of linear operators with respect to various operator topologies.

Perhaps the details of the above talks, and further related results and problems, could be discussed in a working group under the general title Powers and resolvents.

VEČPARAMETRIČNA SPEKTRALNA TEORIJA

PAUL BINDING IN TOMAŽ KOŠIR

V sestavku opišemo večparametrične probleme lastnih vrednosti in predstavimo Atkinsonov algebraični pristop za študij teh problemov.

MULTIPARAMETER SPECTRAL THEORY

Multiparameter eigenvalue problems and Atkinson's algebraic approach to study such problems are described.

One way in which multiparameter eigenvalue problems arise is when the method of separation of variables is used to solve boundary value problems for partial differential equations. Each 'separation constant' gives rise to a different parameter. The resulting equations are simpler boundary value problems for ordinary differential equations, for example of Sturm-Liouville type, that are linked by parameters. Two-parameter problems of this type have been studied since the earliest days of the subject, and the following formulation is, for example, the main object of study in a monograph of Faierman:

$$\frac{d}{dx_i}\left(p_i\left(x_i\right)\frac{dy_i}{dx_i}\right)+\left(\lambda_1 a_i\left(x_i\right)+\lambda_2 b_i\left(x_i\right)-q_i\left(x_i\right)\right)y_i=0,\ i=1,2, \tag{1}$$

where $0 \leq x_i \leq 1$, and boundary conditions are

$$y_i\left(0\right)\cos\alpha_i - p_i\left(0\right)\frac{dy_i}{dx_i}\left(0\right)\sin\alpha_i=0,\ 0\leq\alpha_i<\pi,$$

and

$$y_i\left(1\right)\cos\beta_i - p_i\left(1\right)\frac{dy_i}{dx_i}\left(1\right)\sin\beta_i=0,\ 0<\beta_i\leq\pi,$$

for $i = 1, 2$. These and other problems have motivated the development of multiparameter spectral theory.

In the 1960s Atkinson laid the foundations of abstract multiparameter spectral theory and gave an overview of possible directions for further research. Since then the area has been explored by a number of mathematicians and we mention just a few of them. Analytical aspects were studied by Binding, Browne, Faierman, Sleeman, Turyn and Volkmer, algebraic and geometric aspects by Fainshtein, Grunenfelder, Isaev and Košir, nonlinear problems by Huang, McGhee, Rynne and Shibata, and numerical aspects by Blum, Ji, Müller and Shimasaki. In Slovenia, Vidav's PhD thesis is an

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early contribution to the area. Since then Bohte, Plestenjak, Slivnik and Tomšič have considered numerical aspects of the theory.

One of the main goals of multiparameter spectral theory is to give completeness results for different multiparameter spectral problems. For example, one could try to expand functions defined on the domain of the partial differential equation in terms of Fourier-type series involving the eigenfunctions of the separated (say Sturm-Liouville) equations.

In the abstract theory, the main object studied is the n-tuple of n-parameter pencils

$$W_i(\boldsymbol{\lambda}) = \sum_{j=1}^{n} \lambda_j A_{ij} - A_{i0}, \; i = 1, 2, \ldots, n \; (n \geq 2), \tag{2}$$

also called the multiparameter system. Here A_{ij} are, for all j, linear operators in the Hilbert space H_i. In applications like (1), A_{ij}, $j = 1, 2, \ldots, n$, are multiplication operators and A_{i0} are differential operators. Typically then the A_{ij}, $j \neq 0$ are continuous and the A_{i0} are closed densely defined operators either with compact resolvent or of Fredholm type. In the multiparameter eigenvalue problem we first find n-tuples of complex numbers $\boldsymbol{\lambda}$ such that all the operators $W_i(\boldsymbol{\lambda})$ are singular. This can be considered as a generalization of the ordinary eigenvalue problem.

One fundamental tool of abstract multiparameter spectral theory is a tensor product construction. We consider the tensor product space $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ and certain determinantal operators associated with A_{ij} acting in H. Specifically, Δ_j is (up to a sign) the tensor determinant of the array $[A_{kl}]_{1 \leq k \leq n, 0 \leq l \leq n}$ with j-th column omitted. We limit our interest to so-called nonsingular multiparameter systems when Δ_0 on H is one-to-one. Then the operators $\Gamma_j = \Delta_0^{-1} \Delta_j$ commute and provide a joint spectral decomposition of H. If the A_{ij} are hermitian and $\Delta_0 > 0$ then the eigenvalues are semisimple, and a basis of joint eigenvectors for the Γ_j exists for H. It is important to note that these eigenvectors can in fact be constructed out of (decomposable) tensors of the eigenvectors for the original operators W_i, so the Γ_j do not need constructing explicitly.

In general, completeness requires "joint root subspaces" of the form

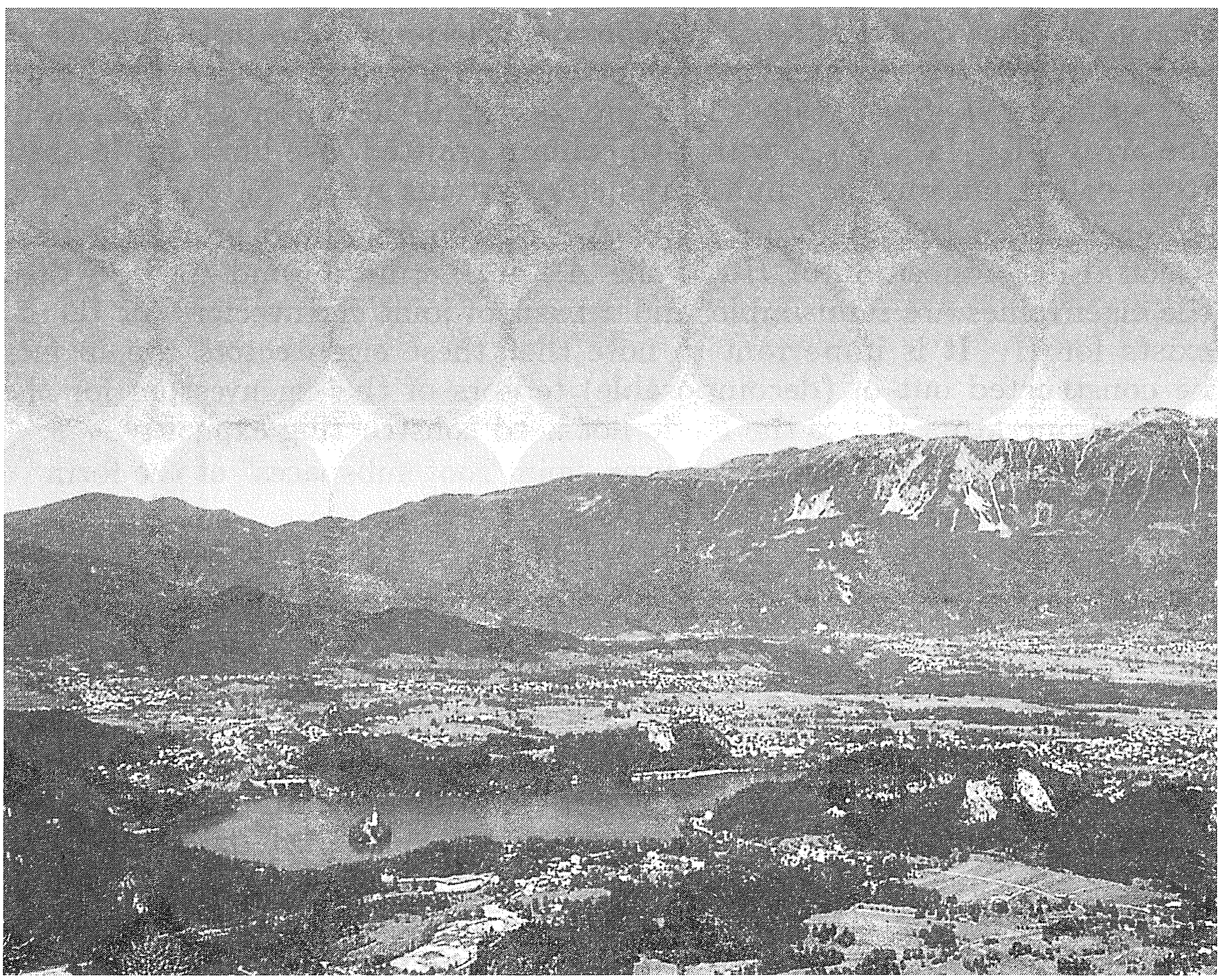
$$R_l(\boldsymbol{\lambda}) = \bigcap_{\sum_{j=1}^{n} \nu_j = l} \mathcal{N}\left(\prod_{j=1}^{n} (\Gamma_j - \lambda_j I)^{\nu_j} \right). \tag{3}$$

Here $\mathcal{N}(A)$ is the nullspace of a linear map A. As before, it is desirable to express completeness in terms of the W_i rather than the Γ_j. In finite dimensions the relative complexity of the two approaches can be gauged from the relation $\dim H = \prod_{i=1}^{n} \dim H_i$. In infinite dimensional examples, when the W_i are ordinary differential operators the Γ_j are partial differential operators. Assuming that the W_i arise from separation of a partial differential operator in the first place, it follows that the main virtue of the technique disappears unless one has completeness statements in terms of the W_i.

Difficulties in proving multiparameter completeness results arise when the eigenvalues are not semisimple, i.e., when root vectors exist.

Various authors constructed root vectors for particular types of eigenvalues and for particular types of multiparameter systems. For instance, Binding described root vectors for real eigenvalues of uniformly-elliptic multiparameter systems, Faierman conjectured the structure of the general root vectors for non-real eigenvalues of the two-parameter spectral problem (1), and Košir described root vectors for nonderogatory and simple eigenvalues of finite-dimensional multiparameter systems. An algebraic construction of the root subspaces (3) was given by Grunenfelder and Košir for general eigenvalues in finite dimensions and for eigenvalues of Fredholm type in infinite dimensions. This construction uses coalgebraic techniques and is in general technically involved. It is expected that one can use special structure of a boundary value problem of type (1), particularly if it is of elliptic type, to simplify the construction and to obtain numerical algorithms suitable for applications.

In the working group we propose to study analytic, algebraic, geometric and numerical aspects of the above described multiparameter eigenvalue problem (2), with special emphasis on elliptic boundary value problems of type (1).



ODREZANI MOMENTNI PROBLEMI: OBSTOJ, ENOLIČNOST IN LOKALIZACIJA NOSILCA UPODOBITVENIH MER

RAÚL E. CURTO

Predstavljeni so odrezani momentni problemi, prirejeni uteženim operatorjem pomika v dveh spremenljivkah v kompleksni ravnini.

TRUNCATED MOMENT PROBLEMS: EXISTENCE, UNIQUENESS, AND LOCALIZATION OF THE SUPPORT OF REPRESENTING MEASURES

We present truncated moment problems associated with 2-variable weighted shifts in the complex plane.

In [6], [13] we succeeded in obtaining a complete solution to the truncated moment problem in case the interpolating measure has compact support in the real line; our main contribution there consisted in bringing to light the notion of *recursiveness*, which was central to our analysis. As we move into several variables, the interpolating measure must be allowed to have support away from the line; one instance of particular interest, associated with 2-variable weighted shifts, is the case of compact support in the complex plane, which we label as the truncated complex moment problem (TCMP).

Let μ be a positive Borel measure on $\mathbb{C}$, assume that $\mathbb{C}[z,\bar{z}] \subseteq L^1(\mu)$ and define $\gamma_{ij} := \int \bar{z}^i z^j d\mu(z,\bar{z})$. Given $p \in \mathbb{C}[z,\bar{z}]$, $p(z,\bar{z}) = \sum_{ij} a_{ij}\bar{z}^i z^j$, we have

$$\begin{aligned} 0 \leq \int \mid p(z,\bar{z}) \mid^2 d\mu(z,\bar{z}) = \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell} \int \bar{z}^{i+\ell} z^{j+k} d\mu(\bar{z},z) = \\ \leq \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell}\gamma_{i+\ell,j+k}. \end{aligned} \tag{1}$$

Observe that $\gamma_{00} > 0$ and $\gamma_{ji} = \overline{\gamma_{ij}}$ for all i, j. To understand the matricial positivity associated with $\gamma := \{\gamma_{ij}\}$, we introduce the following lexicographic order on the rows and columns of infinite matrices: $1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, Z^3, \bar{Z}Z^2, \bar{Z}^2Z, \bar{Z}^3, \ldots$, e.g., the first column is labeled 1, the second column is labeled Z, the third $\bar{Z}$, the fourth Z^2, et cetera; this order corresponds to the graded homogeneous decomposition of $\mathbb{C}[z,\bar{z}]$. For $m, n \geq 0$ let $M[m,n]$ be the $(m+1) \times (n+1)$ block of Toeplitz form whose first row has entries

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given by $\gamma_{mn}, \gamma_{m+1,n-1}, \ldots, \gamma_{m+n,0}$ and whose first column has entries given by $\gamma_{mn}, \gamma_{m-1,n+1}, \ldots, \gamma_{0,n+m}$ (as a consequence, the lower right-hand corner of $M[m,n]$ is γ_{nm}). The matrix $M = M(\gamma)$ is then built as follows:

$$M := \begin{pmatrix} M[0,0] & M[0,1] & M[0,2] & \ldots \\ M[1,0] & M[1,1] & M[1,2] & \ldots \\ M[2,0] & M[2,1] & M[2,2] & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

It is now not hard to see that the above mentioned positivity ((1)) is equivalent to the condition $M \geq 0$, as a quadratic form on $\mathbb{C}^\omega$. Suppose now that we are just given a double-indexed sequence $\gamma \equiv \{\gamma_{ij}\}$ subject to the constraints $\gamma_{00} > 0$ and $\gamma_{ji} = \overline{\gamma_{ij}}$ for all i, j. The classical (complex) *full* moment problem asks for necessary and sufficient conditions on the sequence γ to guarantee the existence of a positive Borel measure μ which interpolates γ, i.e.,

$$\int \bar{z}^i z^j \, d\mu(z, \bar{z}) = \gamma_{ij} \quad (i, j \geq 0). \tag{2}$$

An obvious necessary condition is then $M \geq 0$; this corresponds to the positivity of the Riesz functional $L(p) := \sum_{ij} a_{ij}\gamma_{ij}$ on the cone Σ^2 generated by polynomials of the form $p\bar{p}$. If K is a closed subset of $\mathbb{C}$, the Riesz-Haviland Criterion states that γ admits a representing measure supported on K if and only if $L(p) \geq 0$ for every polynomial p which is nonnegative on K.

With γ, L, M and K as above, suppose there exists a polynomial q such that $K = K_q := \{z \in \mathbb{C} : q(z, \bar{z}) \geq 0\}$. In the presence of a representing measure μ supported on K, the inequality $L_q(p\bar{p}) := L(qp\bar{p}) = \int_K qp\bar{p} \geq 0$ (all $p \in \mathbb{C}[z, \bar{z}]$) must hold, in addition to $L(p\bar{p}) \geq 0$ (all $p \in \mathbb{C}[z, \bar{z}]$). Therefore both conditions are necessary for the existence of a representing measure supported in K. K. Schmüdgen established in [23, Theorem 1] that for K_q compact these two conditions are indeed sufficient, and this is the case also for compact sets K which are *semi-algebraic*, that is, obtained as the intersection of a finite family of K_q's. (For related results, see [23, Corollary 3], [1], [21], [24], [26].)

The *truncated* complex moment problem (TCMP) corresponds to the case when only an *initial segment* of γ is known. Our approach to TCMP follows the strategy we employed to solve the *real* TMP [6]. Indeed, part of the overall strategy can still be carried out, and concrete conditions can be found in a number of fundamental cases.

What we believe must be used now is a combination of a few revealing examples (cf. [7, Chapter 6], [9, Sections 2, 3, 4, and Appendix], [10]) and the interplay between $M(n)$ and $M(n)_q$, a new associated matrix we have introduced in [10].

Theorem 1. *Let $M(n) \geq 0$ and suppose* $\deg q = 2k$ *or* $2k-1$. *There exists* $\operatorname{rank} M(n)$*-atomic representing measure supported in* K_q *if and only if there is some flat extension* $M(n+1)$ *for which* $M_q(n+k) \geq 0$. *In this case, there exists such a representing measure having exactly* $\operatorname{rank} M(n) - \operatorname{rank} M_q(n+k)$ *atoms in* $\mathcal{Z}(q) := \{z \in \mathbb{C} : q(z, \bar{z}) = 0\}$.

M_q keeps track of the location of the support, and this in turn can be used to establish additional constraints when searching for representing measures. In what follows, we list four open problems to be discussed by the research group.

Quadratures and Cubatures. A disc of center a and radius r can be thought of as the *quadrature domain* completely determined by the moments $\gamma_{00} = \pi r^2$, $\gamma_{01} = \pi a r^2$ and $\gamma_{11} = \pi r^2(\frac{r^2}{2} + |a|^2)$, or equivalently, by the moment matrix

$$M(1) = \begin{pmatrix} 1 & a & \bar{a} \\ \bar{a} & \frac{r^2}{2} + |a|^2 & \bar{a}^2 \\ a & a^2 & \frac{r^2}{2} + |a|^2 \end{pmatrix}.$$

Quadrature domains have received ample attention recently, in view of a natural connection with the theory of hyponormal operators with rank-one self-commutator, and with rationally cyclic subnormal operators [15], [16], [17], [18].

Problem 1. *Does the moment matrix $M(n)$ associated with a quadrature domain admit a flat extension, thereby giving rise to a* $\operatorname{rank} M(n)$*-atomic representing measure?*

The study of *minimal* representing measures (those with exactly $\operatorname{rank} M(n)$ atoms) is intimately connected with quadrature problems. For K a closed subset of $\mathbb{R}^n$, w a positive weight function, and d a nonnegative integer, the K-quadrature problem for w of precision d entails finding nodes $x_0, \ldots, x_{M-1} \in K$ and densities $\rho_0, \ldots, \rho_{M-1}$ such that $\int p(x) w(x) dx = \sum_{k=0}^{M-1} \rho_k p(x_k)$ for every polynomial p of total degree d. In [5], [9], and [14], we have applied techniques derived from TCMP to obtain minimal-node solutions for various compact sets in $\mathbb{R}^2$. The problem of *explicitly* computing the nodes and densities of minimal quadrature rules, however, remains largely unsolved, except in special cases (squares and discs, and small values of d; cf. [4]). Our methods circumvent the theory of orthogonal polynomials and considerations of symmetry; instead, the search for $x_0, \ldots, x_{M-1}$ gets restricted to a suitable algebraic variety.

Problem 2. *Find minimal quadrature rules of precision $2n$ $(n > 3)$ for the unit square, unit disc, or equilateral triangle, by building a flat extension of the associated $M(n)$.*

Problem 3. *Let $M(2)$ be a positive moment matrix. Does $M(2)$ always admit a representing measure?*

Our solution of TCMP for flat data was based on the following

Theorem 2 ([7, Theorem 4.7]). *Let M be a finite-rank positive infinite moment matrix. Then M has a unique representing measure, whose support consists of rank$M$ atoms, obtained as the zeros of an associated analytic polynomial.*

Theorem 3 ([7, Theorem 5.4]). *Let $M(n)$ be a flat positive moment matrix (i.e., $\operatorname{rank} M(n) = \operatorname{rank} M(n-1)$). Then $M(n)$ admits a unique flat extension $M(n+1)$.*

A dilation-theoretic approach to cubature. Given a measure ν, a cubature formula of degree $2s-1$ for ν can be thought of as a finitely atomic measure μ such that $\int p \, d\mu = \int p \, d\nu$ for every polynomial p in $\mathcal{P}_{2s-1} := \{p \in \mathbb{C}[x_1, \ldots, x_d] : \deg p \leq 2s-1\}$. If π_{s-1} is the orthogonal projection of $L^2(\nu)$ onto $\mathcal{P}_{s-1}$, and M is the commutative d-tuple of multiplication operators by the coordinates x_j, the compression $\pi_{s-1} M \pi_{s-1}$ is a d-tuple of self-adjoint operators acting on a finite dimensional Hilbert space. In [19], [20], M. Putinar has obtained the following result.

Theorem 4 ([20, Theorem 2.3]). *There exists a bijective correspondence between all finite-rank, cyclic and commutative dilations N of the self-adjoint d-tuple $\pi_{s-1} M \pi_{s-1}$ and triples (m, V, A) consisting of (i) an integer $m \geq s$; (ii) a vector subspace $V \subseteq \mathcal{P}_m$ satisfying $\mathcal{P}_{s-1} \perp (V \cap \mathcal{P}_s)$, $\dim(\mathcal{P}_{m-1}/(V \cap \mathcal{P}_{m-1})) < \dim(\mathcal{P}_m/V)$, and $\mathcal{P}_k V + S = \mathcal{P}_{m+k}$ $(0 \leq k \leq \max(2, m))$, where S is a vector complement of V in $\mathcal{P}_m$; and (iii) a positive operator A on S satisfying $\pi_{s-1} A Q_1 \pi_s = \pi_{s-1}$, where Q_1 is the parallel projection of $\mathcal{P}_{m+1}$ onto S.*

Given a dilation N with cyclic vector $\mathbf{1}$, Putinar builds m and V by looking at the kernel $\mathcal{I}$ of the map $p \mapsto p(N)\mathbf{1}$. Since $\mathcal{I}$ is finite codimensional (because N is finite-rank), the theory of the Hilbert-Samuel polynomial implies that $\mathcal{I}$ is uniquely determined by a positive degree, m, and a subspace V. Consideration of the orthogonal differences $\mathcal{P}_s \ominus (\mathcal{I} \cap \mathcal{P}_s)$ then leads to the operator A in Theorem 4. As an application, one can then obtain an abstract parameterization of all cubature formulas of degree $2s-1$ for a given measure ν. It turns out that such parameterization can be formulated abstractly in terms of certain operator equations, as follows. Let $\mathcal{H}_0, \mathcal{H}_1$ be two finite dimensional Hilbert spaces, let $A, B, C : \mathcal{H}_0 \to \mathcal{H}_1$ be linear operators, and let X_1, Y_1, D be self-adjoint operators acting on $\mathcal{H}_1$. Find and describe all self-adjoint dilations (X, Y) of (X_1, Y_1), acting on the Hilbert space $H = H_1 \oplus H_2$ which satisfy

$$2i[X, Y] = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$X\begin{pmatrix} A \\ 0 \end{pmatrix} + Y\begin{pmatrix} B \\ 0 \end{pmatrix} = \begin{pmatrix} C \\ 0 \end{pmatrix}.$$

The above construction (which describes the space of cubature formulas for a given measure) and the construction in [7, Chapter 4] (which gives an *existence* criterion for solutions to TCMP) have intriguing similarities, which we wish to unravel.

Problem 4. *Establish direct links among: (i) the results in [7, Chapter 4], (ii) the above operator equations, and (iii) the 3-term recurrence relations (associated with Jacobi matrices) studied by Y. Xu [27].*

For the resolution of many of the above problems, some of the tools and techniques that we propose to utilize are derived from our previous work on *joint hyponormality*, which helped us establish the existence of polynomially hyponormal weighted shifts which are not subnormal [12]. For the new problems at hand, we propose to consider suitable combinations of four basic notions:

- positivity for square matrices;
- extendibility of matrices obtained by adding a prescribed number of rows and columns;
- recursiveness; and
- the structure of the real or complex algebraic variety associated to the given moment matrix.

When these four basic ingredients interact in appropriate ways, aided by symbolic manipulation, the result is the construction of concrete algorithms that often describe in detail the space of *all* possible representing measures.

One fundamental idea in the basic construction used in [12] was to extend the intrinsic connection between subnormal operators and classical moment problems in the positive real axis to classes of nearly subnormal operators and moment problems for certain linear functionals not necessarily represented by measures. These techniques also allowed us to obtain a simplification of the main result in [3] for the power moment problem in two dimensions. We would like to further exploit this circle of ideas to obtain further connections between operator theory and classical analysis (cf. [21]).

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POZITIVNI OPERATORJI

ROMAN DRNOVŠEK

Predstavljeni so trije zanimivi odprti problemi o spektru in invariantnih podprostorih za pozitivne operatorje na Banachovih mrežah.

POSITIVE OPERATORS

Three interesting open problems on spectrum and invariant subspaces of positive operators on Banach lattices are presented.

Positive operators on Banach lattices have been studied extensively in the last decades. Their study is a subject of great importance to pure and applied mathematics. The excellent monographs [7], [10], [11], [1], and [8] contain the most important results on this subject.

There are still a lot of open questions on the spectrum and invariant subspaces of positive operators. Let us describe some interesting problems that can be discussed during the informal daily sessions. See [10], [8] and [4] for details about the first two problems.

1. Let T be a positive operator on a Banach lattice E, and let the spectrum $\sigma(T)$ of T contain only the point 1. Denote by I the identity operator on E. The following question is open:

Is it true that $T \geq I$?

This question has affirmative answers in the following cases:

(a) E is finite-dimensional;

(b) 1 is a pole of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$;

(c) T is a lattice homomorphism, i.e., $|Tx| = T|x|$ for all $x \in E$.

2. The *peripheral spectrum* $\sigma_r(T)$ of an operator T on a Banach lattice E is defined by

$$\sigma_r(T) := \sigma(T) \cap \{z \in \mathbb{C} : |z| = r(T)\},$$

where $r(T)$ denotes the spectral radius of T. For positive operator T on E we always have $r(T) \in \sigma_r(T)$. This important fact follows from the inequality

$$|R(\lambda, T)x| \leq R(|\lambda|, T)|x|,$$

which holds for all $x \in E$ and for all λ with $|\lambda| > r(T)$. Moreover, the beautiful theorem of Krein and Rutman asserts that if $r(T)$ is a pole of the resolvent, then $r(T)$ is an eigenvalue of T to which there exists a corresponding positive eigenvector.

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A subset S of $\mathbb{C}$ is said to be *cyclic* if for all $\lambda = |\lambda| w \in S$ it follows that $|\lambda| w^k \in S$ for any integer k. The following question is still unsolved:

Is it true that every positive operator has cyclic peripheral spectrum?

As above, this question has affirmative answers in some special cases:

(a) E is finite-dimesional;

(b) the family $(\lambda - r(T))R(\lambda, T)$ is uniformly bounded for $\lambda > r(T)$.

3. A bounded operator T on a Banach space X is called *idempotent* if $T^2 = T$.

A collection $\mathcal{C}$ of bounded operators on X is said to be *reducible* if there exists a non-trivial invariant closed subspace of X that is invariant under all members of $\mathcal{C}$. Reducibility of multiplicative semigroups of idempotents has recently been studied in [9], [2], [5], and [6]. It was an open question for some years whether every such semigroup is reducible. The answer has been given in [2] where an irreducible semigroup of idempotents on the Hilbert space l^2 is constructed. This example has been recently modified to give the following stronger result (see [3]). Given $K > 1$, there exists an irreducible semigroup of idempotents on l^2 that is norm-bounded by K.

If we restrict our attention to positive idempotents on a Banach lattice, the following question is still open:

Is every multiplicative semigroup of positive idempotents reducible?

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TRANZITIVNE LINEARNE POLGRUPE

LEO LIVSHITS

V sestavku obravnavamo tranzitivnost in nerazcepnost polgrup linearnih operatorjev na končnorazsežnem vektorskem prostoru.

TRANSITIVE LINEAR SEMIGROUPS

Transitivity and irreducibility of semigroups of linear operators on a finite-dimensional vector space are discussed.

Much of motivation for the work described in this abstract came from Operator theoretic realm. As it is often the case there, when trying to answer a general question, one tests a linear algebraic variant first. It is not uncommon for a linear algebraic question to be as hard or harder than the original, but in certain cases, the new question becomes significant in its own right. We present one such account, and shall restrict our attention at present time to the finite-dimensional case.

A proper subspace $\mathcal{M} \neq \{0\}$ of a vector space V (over a field $\mathbb{F}$) is said to be a *(non-trivial) invariant subspace* for a subset S of the algebra $L(\mathcal{V})$ of all linear transformations on V, if $Tx \in \mathcal{M}$ whenever $x \in \mathcal{M}$, $T \in \mathcal{S}$. We call S *irreducible* if it has no non-trivial invariant subspaces. A well-known theorem (named after W. Burnside) states that in the case when $\mathbb{F}$ is algebraically closed and V is finite-dimensional, $L(\mathcal{V})$ has no proper irreducible subalgebras. Burnside's theorem does not extend to infinite-dimensional vector spaces: the subalgebra of $L(\mathcal{V})$ which consists of all transformations of finite rank is irreducible.

If A is a subalgebra of $L(\mathcal{V})$ then, for every $x \in \mathcal{V}$, $\{Ax \mid A \in \mathcal{A}\}(= \mathcal{A}x)$ is an invariant subspace for A. If A is irreducible then $\mathcal{A}x = \{0\}$ or V for every x. Yet $\{x \in \mathcal{V} \mid \mathcal{A}x = \{0\}\}$ is an invariant subspace for A, which shows that $\mathcal{A}x = \mathcal{V}$ for every non-zero x, whenever A is irreducible. It is easy to see that the converse is also true. Expressed in the usual terminology, this states that A is irreducible if and only if it is *transitive*.

The "transitivity" condition $\mathcal{A}x = \mathcal{V} \quad \forall x \neq 0$ means that for every $x, y \in \mathcal{V}$, $x \neq 0$, there exists $A \in \mathcal{A}$ such that $Ax = y$. Obviously the latter form of the definition can be considered to be a particular case of the following general definition:

> A collection $\mathcal{F}$ of functions from a set Ω to a set Δ is *transitive* if for every $x \in \Omega$, $y \in \Delta$ there exists $f \in \mathcal{F}$ such that $f(x) = y$; (F is said to be *sharply transitive* if such f is unique).

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Consider a vector space V over a field $\mathbb{F}$ and let $\mathcal{V}\backslash\{0\}$ play the role of Ω under a subgroup F of the general linear group $GL_{\mathcal{V}}(\mathbb{F})$ of all $\mathbb{F}$-linear bijections of V. Such F is said to be *n-transitive* if for every pair of linearly independent subsets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of V there exists $A \in \mathcal{F}$ such that $A(x_i) = y_i$, $i = 1, \ldots, n$. In case such A is unique we call F *sharply n-transitive.* Finite sharply transitive linear groups have been completely classified by C. Jordan ($n \geq 4$) and H. Zassenhaus [6] ($1 \leq n \leq 3$). There are no infinite sharply n-transitive groups for $n \geq 4$. F. Kalscheuer [3] determined all closed (in Euclidean topology) sharply transitive linear semigroups in the case $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A very active area of research in the last thirty years, transitive linear groups have received a lot of attention due to their importance in finite geometry, where, for example, it is desirable to determine geometries which admit "very transitive" groups of automorphisms.

Keeping with group-theoretic terminology, one can say that an algebra A of linear transformations on a vector space V over a field $\mathbb{F}$ is *n-transitive* if for every linearly independent set $\{x_1, \ldots, x_n\} \subset \mathcal{V}$ and any set $\{y_1, \ldots, y_n\} \subset \mathcal{V}$ there exists $A \in \mathcal{A}$ such that $A(x_i) = y_i$, $i = 1, \ldots, n$. It is marvelous that such a definition is quite redundant: in contrast to both permutation and linear groups, all 2-transitive subalgebras of $L(\mathcal{V})$ are automatically n-transitive, for every $n \in \mathbb{N}$. This is a remarkable theorem of Jacobson [2], and a version of it is true in a more general algebraic setting. (Keep in mind that not every transitive algebra is 2-transitive.) It is also trivial to see that a subset of $L(\mathcal{V})$ is not sharply transitive (under the obvious definition) whenever it is closed under addition.

Linear groups and algebras of linear transformations are two examples of *linear semigroups.* These are the subsets of $L(\mathcal{V})$ closed under composition. Let us define *(n-)transitivity* and *irreducibility* for linear semigroups in the same fashion as this was done for algebras. It is important to notice that no linear group can be transitive as a linear semigroup because its elements cannot send a non-zero vector to zero. To resolve this conflict we shall say that a linear semigroup in $L(\mathcal{V})$ is *n-0-transitive* if for every pair of linearly independent subsets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of V, there exists $A \in \mathcal{S}$ such that $A(x_i) = y_i$, $i = 1, \ldots, n$. It is tempting to define *sharp 0-transitivity* in the obvious way for all linear semigroups, but this definition turns out to be redundant: every sharpy 0-transitive linear semigroup is a linear group possibly together with the zero transformation.

The linear span of a linear semigroup is an algebra possessing the same invariant subspaces as the original semigroup. Every 0-transitive linear semigroup spans a transitive algebra and is hence irreducible. It is easy to see that the converse is false. Jacobson's Theorem does not extend to linear semigroups either: for each $n \in \mathbb{N}$ there exists an n-transitive linear semigroup which is not $(n+1)$-0-transitive. Of course there is no shortage of examples of transitive linear semigroups, apart from linear groups and algebras. Such semigroups shall be our focus. In an attempt to gain better understanding of their structure it is natural to seek out classes of "small" transitive linear semigroups which can be considered basic "building blocks" for constructing many others.

One such class (studied in [1]) consists of the transitive linear semigroups which lack proper transitive left ideals; (a left ideal in a linear semigroup S is a subset $\mathcal{J}$ of S such that $\mathcal{S}\mathcal{J} \subset \mathcal{J}$). We call these *left t-simple*, and the class of transitive left t-simple semigroups contains all minimal transitive semigroups. Ideals play an important role in the theory of transitive linear semigroups. One simple reason is that a non-trivial (two-sided) ideal in a linear semigroup is (0-)transitive if and only if the whole semigroup is (0-)transitive. Semigroups with no non-trivial proper ideals are called *simple* if they do not contain the zero transformation, or *0-simple* if they do.

Let us restrict our attention henceforth to linear semigroups on a finite-dimensional vector space. It turns out that in this case every transitive linear semigroup contains the zero transformation. Hence left t-simple linear semigroups are 0-simple. Furthermore, one can show that every transitive left t-simple linear semigroup S contains a non-zero idempotent, and consequently a primitive idempotent; (a *primitive idempotent* in S is an idempotent minimal with respect to the usual partial order on idempotents in $L(\mathcal{V})$ defined by $e \leq f \Leftrightarrow e = ef = fe$). This makes S an element of a very important collection of (abstract) semigroups: the class of *completely 0-simple* semigroups. There is an extensive representation theory available for this class, including a theorem of D. Rees [4] which classifies all completely 0-simple semigroups (up to an isomorphism) as particular semigroups of linear transformations, under an operation related to a composition. B. M. Schein [5] characterized completely 0-simple semigroups as those isomorphic to simply transitive semigroups of binary relations on a set. The existing representation theory, while providing powerful tools, does not yield the required classification of transitive linear semigroups up to similarity.

Nonetheless, one can make some headway using linear algebraic methods. It is not hard to see that all non-zero elements of a transitive left t-simple linear semigroup S have the same rank, which must divide the dimension of the underlying vector space. If the rank equals the dimension then $\mathcal{S}\backslash\{0\}$ is a 0-transitive group, a case which (for the sake of focus) we treat as "known". Without loss of generality let us concentrate on the case of a vector space $\mathbb{F}^{kr} = \mathbb{F}^r \oplus \mathbb{F}^r \oplus \ldots \oplus \mathbb{F}^r$. The algebra of all linear transformations on this vector space can be interpreted as the block-matrix algebra $\mathbb{M}_k(\mathbb{M}_r(\mathbb{F}))$. Let G be a 0-transitive linear group in $\mathbb{M}_r(\mathbb{F})$ and let

$\Gamma \subset \mathbb{M}_{k\times 1}(\mathcal{G} \cup \{0\})$ be a set containing zero matrix and satisfying two conditions:

1. $\bigcup_{T\in\Gamma} \mathrm{Range}(T) = \mathbb{F}^{kr}$;
2. Distinct elements of Γ have distinct ranges.

Consider the semigroup S in $\mathbb{M}_{1\times k}(\Gamma\mathcal{G})$ (interpreted as a subset of $\mathbb{M}_k(\mathbb{M}_r(\mathbb{F})))$ which consists of those block matrices which have at most one non-zero block column. Then S is transitive left t-simple, and every transitive left t-simple semigroup in $\mathbb{M}_{kr}(\mathbb{F})$ is simultaneously similar to a semigroup of this form. (It is also possible to write down the exact requirements on the relationship between G and Γ which will characterize all minimal transitive left t-simple linear semigroups as well.) This gives complete (and easily understood) characterization of transitive left t-simple semigroups; (the case of minimal such is complicated by a curious algebraic condition, which we shall not state here for the sake of brevity).

Directions for further study

Every transitive left t-simple linear semigroup is clearly minimal transitive. Despite the success in characterizing transitive left t-simple linear semigroups, the same task for minimal transitive semigroups remains unfinished. Perhaps a simpler question is to identify the bigger class of all transitive linear semigroups which lack proper transitive right ideals. This will hopefully lead to the next major step of the inquiry: a classification of all completely 0-simple transitive linear semigroups.

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KONČNORAZSEŽNI PROSTORI Z NEDEFINITNIM SKALARNIM PRODUKTOM: RAZISKOVALNI PROBLEMI

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V sestavku so predstavljeni odprti problemi za unitarne in normalne linearne transformacije na končnorazsežnih vektorskih prostorih z nedefinitnim skalarnim produktom.

FINITE DIMENSIONAL SPACES WITH INDEFINITE SCALAR PRODUCTS: RESEARCH PROBLEMS

Open problems for unitary and normal linear transformations on finite-dimensional vector space with indefinite scalar products are presented.

Let F be either the field of real numbers or the field of complex numbers, and consider a symmetric bilinear (in the real case) or sesquilinear (in the complex case) form $[x, y]$, $x, y \in F^n$. In other words, $[ax + by, z] = a[x, z] + b[y, z]$ and $[y, x] = \overline{[x, y]}$ for all $x, y, z \in F^n$ and all scalars $a, b \in F$. In the real case, we consider also skew bilinear forms, i.e., those for which the second condition is replaced by $[x, y] = -[y, x]$ for all $x, y \in \mathbb{R}$. These forms are often called *indefinite scalar* (or *inner*) *products*, to emphasize connections with the standard (positive definite) *scalar* (or *inner*) *products* which are characterized by the additional requirement that $[x, x] \geq 0$ for all $x \in F^n$ and $[x, x] = 0$ if and only if $x = 0$. An indefinite scalar product $[\cdot, \cdot]$ is called *regular* if $[x, y] = 0$ for all $y \in F^n$ implies $x = 0$.

It is of interest to study structures of linear transformations on F^n that respect certain properties described in terms of $[\cdot, \cdot]$. An interesting class of problems involves study in the indefinite scalar products context of results that are well-known and widely used for positive definite scalar products. Recently, there has been renewed interest in indefinite scalar products, and many results along the lines just described have been obtained, in particular, concerning normal transformations (see [9], [10], [12], [13], [14] [17]), polar decompositions see (see [3], [4], [5]), singular values [6], numerical ranges (see [16], [15]), plus transformations [18], contractions and inertia (see [1], [2]), etc. Many interesting problems in this area have not been studied yet. Several open problems are listed below. As it is often done in linear algebra, the problems and results will be stated in terms of matrices rather than linear transformations.

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Fix an indefinite scalar product $[\cdot,\cdot]$. A matrix $A^* \in F^{n\times n}$ is called *adjoint* of $A \in F^{n\times n}$ if $[Ax, y] = [x, A^*y]$ for all $x, y \in F^n$. Although A^* is uniquely defined only when the indefinite scalar product is regular, nevertheless the concepts of *selfadjoint* and *unitary* matrices A may be defined even for non-regular indefinite scalar products by the equalities $[Ax, y] = [x, Ay]$ for all $x, y \in F^n$, and $[Ax, Ay] = [x, y]$ for all $x, y \in F^n$, respectively. Canonical forms of selfadjoint matrices are well-known, implicitly they are found in [19], for example.

Problem 1. *Develop canonical forms for unitary matrices, for not necessarily regular indefinite scalar products.*

For regular complex sesquilinear indefinite scalar products, several canonical forms of unitary matrices have been developed; see Chapter I.4 in [8], [10]. Canonical forms of symplectic (i.e., unitary with respect to a regular real skew symmetric indefinite scalar product) *matrix pencils* $\lambda A + B$ are given in [20].

If $[\cdot,\cdot]$ is regular, then A is called *normal* if $AA^* = A^*A$. For non-regular indefinite scalar products, the notion of normal matrices may be defined as well: Let $[x, y] = (Sx, y)$, $x, y \in F^n$, where S is the real symmetric (or complex Hermitian, or real skew symmetric, as the case may be) matrix that determines the indefinite scalar product. Then A is called *normal* if $S^+A^HSA = AS^+A^HS$, where A^H stands for the conjugate transpose of A, and S^+ is the Moore-Penrose inverse of S.

So far, indecomposable normal matrices and the corresponding canonical forms have been described only in the cases of regular real symmetric and complex sesquilinear $[\cdot,\cdot]$ when S has at most 2 negative eigenvalues (see [10], [12], [13]). The problem of characterizing indecomposable normal matrices in general seems to be intractable.

Problem 2. *Obtain canonical forms for some other classes of normal matrices, in particular, involving non-regular indefinite scalar products.*

In the standard positive definite scalar product, the normality of a matrix can be characterized in many ways [11], [7]. It would be of interest to sort out these ways for indefinite scalar products. Very likely, most of them will fail to be characteristic of normality. For example, the set of matrices A such that A^* is a polynomial of A, is a proper subset of the set of normal matrices.

A factorization $X = UA$, where U is unitary and A is selfadjoint, is called a *polar decomposition* of X. (Note that in contrast to the standard definition we do not require that A be positive semidefinite.) Polar decompositions have been studied in detail for regular real symmetric and complex sesquilinear indefinite scalar products in [3], [4], [5].

Problem 3. *Characterize those normal matrices that admit polar decompositions.*

In connection with this problem, note that not every matrix admits a polar decomposition. It was proved in [3] that every normal matrix admits a polar decomposition if the indefinite scalar product is real symmetric or complex sesquilinear and the corresponding matrix S has only one negative eigenvalue. It is conjectured that every normal matrix admits polar decompositions.

One proves easily that if $X = UA$ is polar decomposition and U and A commute, then X is normal. The converse holds for the positive definite scalar product.

Problem 4. *Is the converse true in indefinite scalar products? In other words, if X is normal and has a polar decomposition $X = UA$, must U and A commute (for some choice of the polar decomposition)?*

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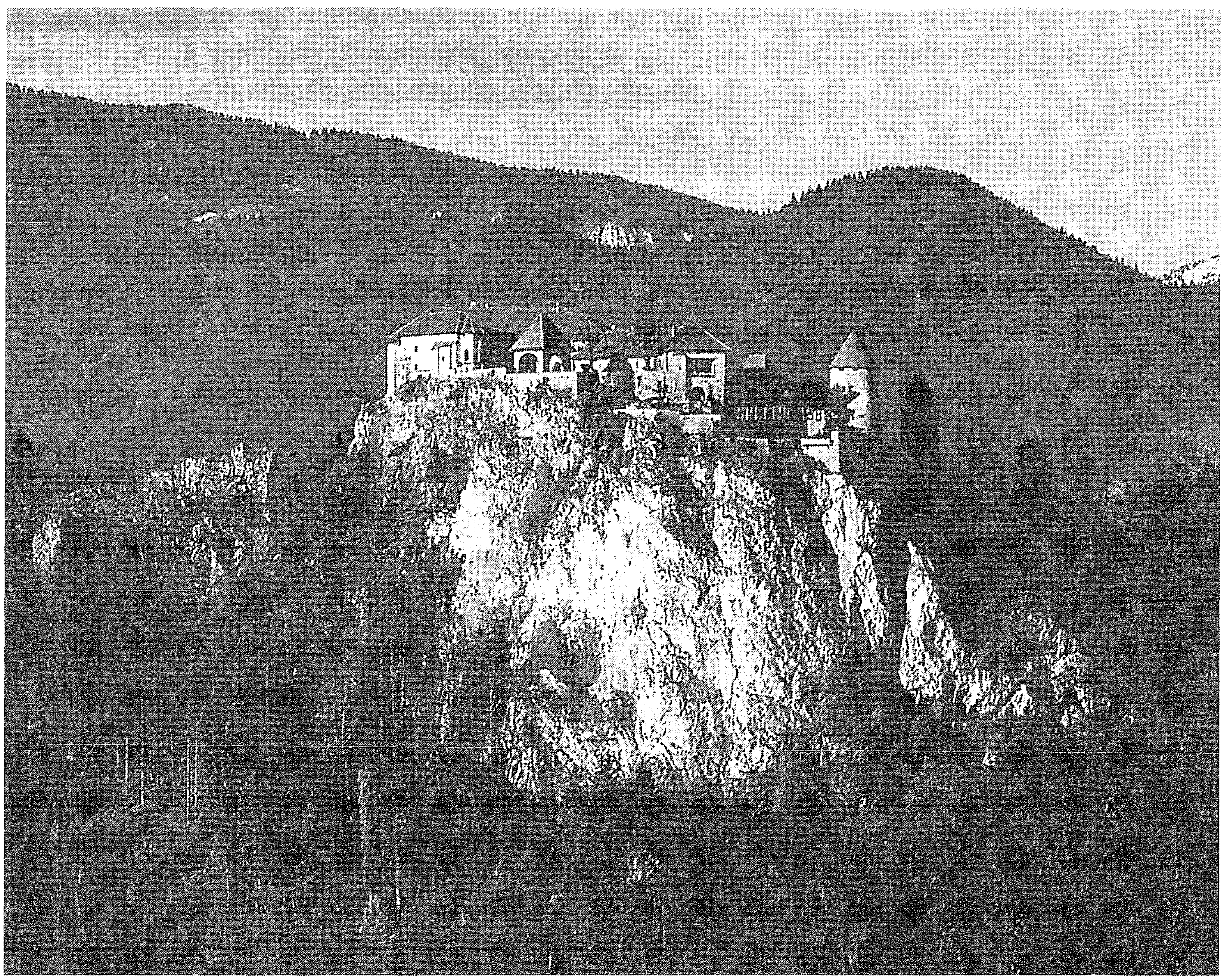

LINEARNE PRESLIKAVE, KI OHRANJAJO OBRNLJIVOST

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Predstavljen je problem karakterizacije linearnih preslikav, ki ohranjajo obrnljivost.

LINEAR MAPS PRESERVING INVERTIBILITY

The problem of characterization of linear maps preserving invertibility is discussed.

Let R and R' be two rings with identities 1 and $1'$, respectively. A map $\phi : R \to R'$ is called *unital* if $\phi(1) = 1'$ and it is called *invertibility preserving* if $\phi(x)$ is invertible in R' for every invertible $x \in R$. What are examples of additive maps having these two properties? Clearly, every isomorphism of rings, as well as every anti-isomorphism (that is, a bijective additive map $\phi : R \to R'$ satisfying $\phi(xy) = \phi(y)\phi(x)$, $x, y \in R$) is a unital invertibility preserving map.

Isomorphisms and anti-isomorphisms are special examples of Jordan isomorphisms. An additive map $\phi : R \to R'$ is called a Jordan homomorphism if $\phi(x^2) = \phi(x)^2$, $x \in R$; if it is also bijective, then it is called a Jordan isomorphism.

Isomorphisms and anti-isomorphisms are basic, but not the only examples of Jordan isomorphisms. Indeed, take any isomorphic noncommutative rings R_1 and R_1', and anti-isomorphic noncommutative rings R_2 and R_2'. Let R and R' be direct sums $R = R_1 \oplus R_2$ and $R' = R_1' \oplus R_2'$ where the operations are defined componentwise. Then $\phi : R \to R'$ defined by $\phi(a_1 \oplus a_2) = \varphi_1(a_1) \oplus \varphi_2(a_2)$, where $\varphi_1 : R_1 \to R_1'$ is an isomorphism and $\varphi_2 : R_2 \to R_2'$ is an anti-isomorphism, is a Jordan isomorphism which is neither an isomorphism nor an anti-isomorphism. On the other hand, a well–known Herstein's result [I. N. Herstein, *Jordan homomorphisms*, Trans. Amer. Math. Soc. **81** (1956), 331–341] on Jordan homomorphisms implies that examples of Jordan isomorphisms different from isomorphisms and anti-isomorphisms can be produced only if one of the rings contains two nonzero ideals whose product is zero.

Now let R be an arbitrary unital ring and R' be any unital ring such that $2x \neq 0$ for any nonzero $x \in R'$. Then every Jordan isomorphism $\phi : R \to R'$ is a unital invertibility preserving map [A. R. Sourour, *Invertibility preserving linear maps on $\mathcal{L}(X)$*, Trans. Amer. Math. Soc. **348** (1996), 13–30]. The proof of this statement is elementary and not very difficult.

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Kaplansky [I. Kaplansky, *Algebraic and analytic aspects of operator algebras*, Regional Conference Series in Mathematics 1, Amer. Math. Soc., Providence, 1970] asked when the converse is true. More precisely, which conditions on R and R' imply that every unital additive map $\phi : R \to R'$ preserving invertibility is a Jordan homomorphism? As one may expect, solving this problem is a much more difficult task even for some simple classes of rings.

Most of the work on this problem was done by mathematicians working in functional analysis. Therefore, we will consider linear maps on algebras rather than additive maps on rings. Moreover, we restrict our attention to the case when $\mathcal{A}$ and $\mathcal{B}$ are unital complex Banach algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is a linear invertibility preserving map. Also, without loss of generality we assume that ϕ is unital for otherwise we could consider the map $a \mapsto \phi(1)^{-1}\phi(a)$. The goal is to find reasonable conditions implying that ϕ is a Jordan homomorphism.

For an element $a \in \mathcal{A}$ we define the spectrum $\sigma(a)$ of a as the set of all complex numbers λ such that $\lambda \cdot 1 - a$ is not invertible. Hence, in the case that $\mathcal{A} = M_n$, the algebra of all $n \times n$ complex matrices, the spectrum of a matrix is the set of all its eigenvalues. Note that the condition that a unital linear map ϕ preserves invertibility can be reformulated as $\sigma(\phi(a)) \subset \sigma(a)$ for every $a \in \mathcal{A}$. So, we can ask a question that is somewhat easier than the original Kaplansky's problem: when must a spectrum preserving unital linear map between two Banach algebras be a Jordan homomorphism? Here, of course, by a spectrum preserving map we mean a map $\phi : \mathcal{A} \to \mathcal{B}$ satisfying $\sigma(\phi(a)) = \sigma(a)$, $a \in \mathcal{A}$. A lot of work has been done also on related problems of characterizing linear maps preserving certain spectral properties.

The following conjecture seems to be reasonable: Let $\mathcal{A}$ and $\mathcal{B}$ be semi-simple Banach algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a unital bijective linear map preserving invertibility. It is then conjectured that ϕ must be a Jordan isomorphism. This conjecture was formulated (although not always in this generality) by many mathematicians working on this problem; as far as we know, however, Aupetit was the first one.

It seems that at the present no techniques are available to handle the general case. In particular, the problem is still open for C^*-algebras and even for von Neumann algebras as was pointed out by Harris and Kadison [L. A. Harris, R. V. Kadison, *Affine mappings of invertible operators*, Proc. Amer. Math. Soc. **124** (1996), 2415–2422].

PROBLEMI RAZCEPNOSTI ZA DRUŽINE OPERATORJEV

HEYDAR RADJAVI

V sestavku so opisani problemi razcepnosti za družine linearnih operatorjev v primerih, ko ima družina še kako dodatno strukturo (npr. je polgrupa, Liejeva ali Jordanova algebra itd.) in za operatorje velja še kaka dodatna lastnost (npr. permutabilnost, submultiplikativnost ali sublinearnost spektra, itd.).

REDUCIBILITY PROBLEMS FOR OPERATOR FAMILIES

Families of linear operators with additional structure (e.g. the family is a semigroup, or a Lie or Jordan algebra, etc.) and additional properties (e.g. permutability, submultiplicativity or sublinearity of spectra, etc.) are considered and reducibility problems for such families are discussed.

1. Introduction

The format of the workshop will be very similar to the one we had in 1996 at Bled, which proved quite successful. After one or two introductory formal lectures, which introduce, discuss and elaborate on some of the problems mentioned below, we meet at informal daily sessions to go more deeply into a small number of the problems, work on them both individually and in groups, and exchange ideas and partial solutions. This informal setting allows discussing any of the problems that are of greater interest to the participants, and even going back-and-forth between two or three problems as the general mood of the participants dictates.

Many of the young mathematicians and graduate students who took part in our last Bled workshop found it very stimulating and rewarding. We should mention that one of the topics discussed, namely that of commutators of rank one, was so thoroughly studied during the workshop period that a substantial paper, authored by seven of the participants, came out of it. It was published in the Journal of Functional Analysis [1].

2. Problems to be studied

A family of (linear) operators on a vector space V over an algebraically closed field is said to be *irreducible* if no non-trivial subspace of V is invariant under (every member of) $\mathcal{F}$. A family $\mathcal{F}$ is called *transitive* if for every $x \neq 0$ and y in V, there is a member A of $\mathcal{F}$ with $Ax = y$. It is easy to see that if $\mathcal{F}$ is an algebra (that is, if it is closed under linear combinations and products), then $\mathcal{F}$ is transitive if and only if it is irreducible. Perhaps, the most well-known result along these lines is the classical Burnside's Theorem that the only transitive algebra of operators on a finite-dimensional V is the full algebra $\mathcal{L}(V)$ of all operators on V.

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If $\mathcal{F}$ is not an algebra, e.g., when it is just a multiplicative or additive semigroup, or if it is a Lie or Jordan algebra, transitivity implies irreducibility, but not necessarily conversely. Both problems are of interest in pure and applied mathematics for different families of operators.

Several of the participants in the workshop have worked in various areas related to these problems. For example, the paper [4] considers the Jordan-algebra analogue of the Burnside's Theorem and gives a satisfactory solution. The Lie-algebra analogue is still open. Transitive matrix groups have a long history. The corresponding problem for semigroups have been discussed recently in [2]; another variation, specializing to vectors and matrices with non-negative entries is the subject of another recent paper [6].

Given a structured family, say a multiplicative semigroup $\mathcal{S}$, there have been many studies in the last two decades on what spectral conditions on the members of $\mathcal{S}$ result in reducibility or simultaneous triangularizability of $\mathcal{S}$. For example, the papers [3], [5], and [7] discuss the effect of submultiplicative and permutable spectrum or spectral radius on reducibility. Spectrum is said to be submultiplicative on $\mathcal{S}$ if $\sigma(AB) \subseteq \sigma(A)\sigma(B)$ for all A and B in $\mathcal{S}$. ($\sigma(T)$ denotes the spectrum of T.) Permutability means $\sigma(ABC) = \sigma(BAC)$ for all A, B, C in $\mathcal{S}$. In the complex case, spectral radius ρ, is called submultiplicative if $\rho(AB) \leq \rho(A)\rho(B)$ for all A and B in $\mathcal{S}$. A sample result is that permutable spectrum implies reducibility. A sample unsolved problem that could be discussed in the current workshop: does there exist an irreducible matrix group of order 2^k with submultiplicative spectrum? (For orders other than 2^k, the answer is yes.)

There are many more problems in this area that can be discussed. Some structure problems even for matrix groups with spectral conditions imposed on them still remain open. We know, for instance, that an irreducible group of complex matrices with submultiplicative spectrum is essentially finite (it is contained in $\mathbb{C}\mathcal{G}$ with $\mathcal{G}$ a finite group) and nilpotent, but their general structure is not completely known.

There are also interesting topological versions of these problems. It is known [7] that on semigroups of compact operators on a Hilbert space, σ is permutable if and only if it is submultiplicative. Whether this is true or not for general semigroups of operators remains open.

As a final example, we mention the properties of sublinearity of spectrum on a semigroup: σ is said to be sublinear on a semigroup $\mathcal{S}$ if $\sigma(A + \lambda B) \subseteq \sigma(A) + \lambda\sigma(B)$ for all pairs A, B in $\mathcal{S}$. This condition does imply simultaneous triangularizability [8]. There are unsolved related problems that could be discussed. Sample: if σ is subadditive on $\mathcal{S}$, i.e., $\sigma(A + B) \subseteq \sigma(A) + \sigma(B)$ for all A and B, can $\mathcal{S}$ be irreducible?

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